# On the inequalities defining the entanglement space of 2-qubits

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**Abstract:** The issue of description of the *entanglement space* $\mathcal{E}_2$, i.e., the orbit space $\mathfrak{P}_+/\mathrm{G}$, where $\mathfrak{P}_+$ - the space of mixed states of pair of qubits, $\mathrm{G} = \mathrm{U}(2) \otimes \mathrm{U}(2)$ - the group of so-called local unitary transformations, is discussed. Within the geometrical invariant theory, using the integrity basis for the ring of G-invariant polynomials, the derivation of equations and inequalities that determine the entanglement space $\mathcal{E}_2$ are outlined.

**•Quantum non-localities and orbit space •** A motivation to study the orbits space $\mathfrak{P}_+/\mathrm{G}$ for $d$-dimensional $r$-partite quantum system is as follows. A state $\varrho \in \mathfrak{P}_+$, characterizing a composite quantum system is an element of the tensor product of Hilbert-Schmidt spaces of operators corresponding to each $r$ individual subsystem. In accordance with a fixed factorization $d = n_1 \times n_2 \times \cdots \times n_r$, the Local Unitary (LU) group, $\mathrm{G} = \mathrm{U}(n_1) \otimes \cdots \otimes \mathrm{U}(n_r)$ acts on $\mathfrak{P}_+$ in non-transitive way. This circumstance causes a stratification of $\mathfrak{P}_+$, reflecting a diversity of non-local properties the system exposes. Classes of the equivalence with respect to the LU transformations form the so-called entanglement space, the factor space:

$$\mathcal{E} = \frac{\text{Space of states}}{\text{Group of LU transformations}}\,.$$

Thus characterization and classification of a quantum system non-locality reduces mainly to a classical mathematical problem - description of the orbit space of compact Lie groups.

•**Recipe for the orbit space description** • The orbit space of a compact Lie group action on a linear space can be described in the framework of the invariant theory within the direction initiated by Processi and Schwarz [1, 2].

Consider the compact Lie group G acting linearly on the real $d$-dimensional vector space $V$ and let $\mathbb{R}[V]^{\mathrm{G}}$ is the corresponding ring of the G-invariant polynomials on $V$. Assume $\mathcal{P} = (p_1, p_2, \dots, p_q)$ is a set of homogeneous polynomials that form the integrity basis, $\mathbb{R}[x_1, x_2, \dots, x_d]^{\mathrm{G}} = \mathbb{R}[p_1, p_2, \dots, p_q]$. Elements of the integrity basis define the polynomial mapping:

$$p : \qquad V \to \mathbb{R}^q ; \qquad (x_1, x_2, \dots, x_d) \to (p_1, p_2, \dots, p_q) .$$

Since $p$ is constant on the orbits of G it induces a homeomorphism of the orbit space $V/G$ and the image $X$ of $p$-mapping; $V/G \simeq X$ [3]. In order to describe $X$ in terms of $\mathcal{P}$ uniquely, it is necessary to take into account the *syzygy ideal* of $\mathcal{P}$, i.e.,

$$I_{\mathcal{P}} = \{ h \in \mathbb{R}[y_1, y_2, \dots, y_q] : h(p_1, p_2, \dots, p_q) = 0 \,, \text{ in } \mathbb{R}[V] \,\}.$$

Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of $I_{\mathcal{P}}$ , then $Z$ is algebraic subset of $\mathbb{R}^q$ such that $X \subseteq Z$ . Denoting by $\mathbb{R}[Z]$ the restriction of $\mathbb{R}[y_1, y_2, \dots, y_q]$ to $Z$ one can easily verify that $\mathbb{R}[Z]$ is isomorphic to the quotient $\mathbb{R}[y_1, y_2, \dots, y_q]/I_{\mathcal{P}}$ and thus $\mathbb{R}[Z] \simeq \mathbb{R}[V]^{\mathrm{G}}$ . Therefore the subset $Z$ essentially is determined by $\mathbb{R}[V]^{\mathrm{G}}$, but to describe $X$ the further steps are required. According to [1, 2] the necessary information on $X$ is encoded in the structure of $q \times q$ matrix with elements given by the inner products of gradients, $\mathrm{grad}(p_i)$ :

$$||\mathrm{Grad}||_{ij} = \left(\mathrm{grad}\,(p_i)\,, \mathrm{grad}\,(p_j)\right) \,.$$

Summarizing these observations, the orbit space is identified with the semi-algebraic variety, defined as points, satisfying two conditions:

a) $z \in Z$, where $Z$ is the surface defined by the syzygy ideal for the integrity basis of $\mathbb{R}[V]^{\mathrm{G}}$;

b) $\mathrm{Grad}(z) \geqslant 0$ .

**•Describing the entanglement space $\mathcal{E}_2$ •** The general scheme sketched above has been applied to the analyzes of a 4-dimensional bipartite quantum system with partition, $n_1 = n_2 = 2$, i.e., a pair of qubits.

To make Procesi-Schwarz method applicable we linearize at first the adjoint action of $\mathrm{U}(2) \otimes \mathrm{U}(2)$ group on the space $\mathcal{H}_{4\times 4}$ of $4 \times 4$ Hermitian matrices:

$$(\mathrm{Ad}\, g\,)\varrho = g\,\varrho\, g^{-1}\,, \qquad g \in \mathrm{U}(2) \otimes \mathrm{U}(2)\,, \tag{1}$$

by the mapping $\mathcal{H}_{4\times 4} \to \mathbb{R}^{16}; \varrho \to \boldsymbol{v} = (v_1, v_2, \dots, v_{16})$ and considering on $\mathbb{R}^{16}$ the linear representation

$$\boldsymbol{v}' = L\boldsymbol{v}\,, \qquad L \in \mathrm{U}(2) \otimes \mathrm{U}(2) \otimes \overline{\mathrm{U}(2) \otimes \mathrm{U}(2)}\,,$$

where a line over expression means the complex conjugation. Further using the integrity basis for $\mathbb{R}[\boldsymbol{v}]^{\mathrm{U}(2)\otimes\mathrm{U}(2)}$, suggested in [4]-[7] one can pass to the analysis of the semi-positivity of the Grad- matrix and determine the set of inequalities defining the orbit space $\mathbb{R}^{16}/\mathrm{U}(2) \otimes \mathrm{U}(2)$. However, this is not the end of a story. The orbit space defined in this manner is not the space of entanglement, namely $\mathcal{E}_2 \subseteq \mathbb{R}^{16}/\mathrm{U}(2) \otimes \mathrm{U}(2)$. Indeed, due to the non-negativity of density matrices the space of physical states is $\mathfrak{P}_+ \subset \mathbb{R}^{15}$ defining by a further set of constraints on elements of integrity basis (see e.g. [7]). Concluding it is worth to stress that analysis of the relevant geometry of $\mathcal{E}_2$, determining via a complete set of polynomial inequalities in LU invariants, including both, mentioned here, as well as arising from the semi-positivity conditions on the density matrix of 2 -qubits, represents a non-trivial mathematical problem and has highly important consequences for quantum information theory and quantum computing.

**• Computational issues •** To derive the inequalities in the LU invariants determining the orbit space $\mathbb{R}^{16}/\mathrm{U}(2) \otimes \mathrm{U}(2)$, one has first to express the entries of Grad-matrix in terms of the invariants and then compute its Smith normal form. For the last computation we are going to try recent algorithms [8] and their implementation in Maple. Unlike all previously known algorithms for reduction of a matrix to the Smith normal form, the algorithms of paper [8] may work when the entries of a matrix are multivariate polynomials. The ring of such polynomials is not Euclidean (i.e., not principal ideal) domain that is at the basis of all other algorithms.

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