Bivariate Dimension Quasi-polynomials of Difference-Differential Field Extensions with Weighted Basic Operators

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We prove the existence and determine some invariants of a Hilbert-type bivariate quasi-polynomial associated with a difference-differential field extension with weighted basic derivations and translations. We show that such a quasi-polynomial can be expressed in terms of univariate Ehrhart quasi-polynomials of rational conic polytopes.

1. Preliminaries

Let K be a difference-differential field of zero characteristic with basic sets of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ and automorphisms $\sigma = \{\alpha_1, \dots, \alpha_n\}$ (any two mappings from the set $\Delta \cup \sigma$ commute) and let every δ_i, $1 \le i \le m$ (respectively, every α_j, $1 \le j \le n$), be assigned a positive integer weight v_i (respectively, w_j). Let Λ be the free commutative semigroup generated by the set $\Delta \cup \sigma$ whose elements are written as power products $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ $(k_i, l_j \in \mathbb{N})$.

We define the orders of λ with respect to the sets Δ and σ (and with respect to the given weights) as $\text{ord}_\Delta \lambda = \sum_{i=1}^m v_i k_i$ and $\text{ord}_\sigma \lambda = \sum_{j=1}^n w_j l_j$, respectively, and set $\Lambda_{V,W}(r,s) = \{\lambda \in \Lambda \,|\, \text{ord}_\Delta \lambda \le r, \text{ord}_\sigma \lambda \le s\}$ for all $r, s \in \mathbb{N}$.

In what follows, we will use the prefix Δ-σ- instead of the adjective "difference-differential". If $\eta = \{\eta_1, \dots, \eta_q\}$ is a finite subset of a Δ-σ-overfield of K, we write $K\langle \eta_1, \dots, \eta_q \rangle$ for the Δ-σ-field extension of K generated by the set η. (As a field, it coincides with $K(\{\lambda(\eta_i) \,|\, \lambda \in \Lambda, 1 \le i \le q\})$.)

2. Dimension quasi-polynomials of subsets of $\mathbb{N}^p$

A function $f : \mathbb{Z} \to \mathbb{Q}$ is called a (univariate) *quasi-polynomial* of period q if there exist q polynomials $g_i(x) \in \mathbb{Q}[x]$ $(0 \le i \le q-1)$ such that $f(n) = g_i(n)$ whenever $n \in \mathbb{Z}$ and $n \equiv i \,(mod\, q)$.

An equivalent way of introducing quasi-polynomials is as follows.

A *rational periodic number* $U(n)$ is a function $U : \mathbb{Z} \to \mathbb{Q}$ with the property that there exists (a period) $q \in \mathbb{N}$ such that $U(n) = U(n')$ whenever $n \equiv n' \,(mod\, q)$.

A rational periodic number can be represented by a list of q its possible values: $U(n) = [a_0, \dots, a_{q-1}]_n$. For example, $U(n) = \left[\frac{1}{2}, \frac{3}{4}, 1\right]_n$ is a periodic number with period 3 such that $U(n) = \frac{1}{2}$ if $n \equiv 0 \,(mod\, 3)$, $U(n) = \frac{3}{4}$ if $n \equiv 1 \,(mod\, 3)$, and $U(n) = 1$ if $n \equiv 2 \,(mod\, 3)$.

With the above notation, a (univariate) *quasi-polynomial of degree d* is a function $f : \mathbb{Z} \to \mathbb{Q}$ such that

$$f(n) = c_d(n)n^d + \cdots + c_1(n)n + c_0(n)$$

where $c_i(n)$'s are rational periodic numbers and $c_d(n) \neq 0$ for at least one $n \in \mathbb{Z}$.

One of the main applications of the theory of quasi-polynomials is its application to the counting of integer points in polytopes.

Recall that a *rational polytope* in $\mathbb{R}^d$ is the convex hall of finitely many points (vertices) in $\mathbb{Q}^d$ or, equivalently, the set of solutions of a finite system of linear inequalities $A\mathbf{x} \leq \mathbf{b}$, where A is an $l \times d$-matrix with integer entries (l is a positive integer) and $\mathbf{b} \in \mathbb{Z}^l$, provided that the solution set is bounded.
Let $P \subseteq \mathbb{R}^d$ be a rational polytope. In what follows, we assume that P has dimension d, that is, P is not contained in a proper affine subspace of $\mathbb{R}^d$. Then a polytope $rP = \{r\mathbf{x} \,|\, \mathbf{x} \in P\}$ ($r \in \mathbb{N}$) is called the rth *dilate* of P. (Clearly, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are all vertices of P, then rP is the convex hall of $r\mathbf{v}_1, \ldots, r\mathbf{v}_k$.) The number of integer points (that is, points with integer coordinates) in rP is denoted by $L(P, r)$. The following result is due to E. Ehrhart, see [3].

Theorem 1 *$L(P, r)$ is a degree d quasi-polynomial of r whose leading coefficient is equal to the Euclidean volume of P.*

The main tools for the computation of Ehrhart quasi-polynomials are Alexander Barvinok's polynomial time algorithm and its modifications, see [1] and [2].

Let $\mathbf{p} = (p_1, \ldots, p_r)$ be an r-dimensional parameter vector. An *r-dimensional periodic number $U(\mathbf{p})$ on $p_1, \ldots, p_r$* is a function $U : \mathbb{Z}^r \to \mathbb{Q}$ such that there exists $\mathbf{q} = (q_1, \ldots, q_r) \in \mathbb{N}^r$ with the property that $U(p_1, \ldots, p_r) = U(p'_1, \ldots, p'_r)$ whenever $p_i \equiv p'_i (mod\, q_i)$, $1 \leq i \leq r$. The least common multiple of all q_i is called a *period* of U. Say, $[[1, \frac{3}{2}]_{p_2}, [0, \frac{3}{4}]_{p_2}, [-1, \frac{1}{5}]_{p_2}]_{p_1}$ is a 2-periodic number with period 6 ($\mathbf{q} = (3, 2)$).

A polynomial in r variables $p_1, \ldots, p_r$, where each coefficient is a multidimensional periodic number on a subset of $\{p_1, \ldots, p_r\}$, is called a *multivariate quasi-polynomial* (in $p_1, \ldots, p_r$). Its *period* is defined as the least common multiple of the periods of the coefficients.

Let $m, n \in \mathbb{N}$, $A \subseteq \mathbb{N}^{m+n}$ and $X_A = \{\mathbf{x} = (x_1, \ldots, x_{m+n}) \,|\, \mathbf{x}$ is not greater than or equal to any $\mathbf{a} \in A$ with respect to the product order $<_P$ on $\mathbb{N}^{m+n}\}$. (Recall that $(a_1, \ldots, a_{m+n}) <_P (x_1, \ldots, x_{m+n})$ if $a_i < x_i$ for $i = 1, \ldots, m+n$.)

Let us fix two sets of positive integers $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ ("weights") and define the orders of an $(m+n)$-tuple $\mathbf{a} = (a_1, \ldots, a_{m+n}) \in \mathbb{N}$ with respect to these sets as $\mathrm{ord}_V\,\mathbf{a} = \sum_{i=1}^{m} v_i a_i$ and $\mathrm{ord}_W\,\mathbf{a} = \sum_{i=m+1}^{m+n} w_i a_i$, respectively. Furthermore, for any set $A \subseteq \mathbb{N}^{m+n}$ and for any $r,s \in \mathbb{N}$, let

$$A(r,s) = \{\mathbf{a} \in A \,|\, \mathrm{ord}_V\,\mathbf{a} \le r,\, \mathrm{ord}_W\,\mathbf{a} \le s\}.$$

Theorem 2 *With the above notation, there exists a bivariate quasi-polynomial $\phi_{V,W}(t_1,t_2)$ such that*
(i) $\phi_{V,W}(r,s) = \mathrm{Card}\,X_A(r,s)$ *for all sufficiently large $(r,s) \in \mathbb{N}^2$. (It means that there is $(r_0,s_0) \in \mathbb{N}^2$ such that the equality holds for all integers $r \ge r_0$, $s \ge s_0$.)*
(ii) $\deg_{t_1}\phi_{V,W} \le m$ *and* $\deg_{t_2}\phi_{V,W} \le n$.
(iii) $\deg\phi_{V,W} = m+n$ *if and only if* $A = \emptyset$
(iv) $\phi_{V,W}(t_1,t_2) = 0$ *if and only if* $(0,\dots,0) \in A$.

3. The main result

In what follows we keep the notation of section 1.

Theorem 3 *Let K be a Δ-σ-field and let $L = K\langle\eta_1,\dots,\eta_q\rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1,\dots,\eta_q\}$. For any $r,s \in \mathbb{N}$, let $L_{r,s} = K(\{\lambda(\eta_i)\,|\,\lambda \in \Lambda_{V,W}(r,s),\, 1 \le i \le q\})$. Then there exists a bivariate quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1,t_2)$ such that*

(i) $\Phi_{\eta|K}^{(V,W)}(r,s) = \mathrm{tr.deg}_K L_{rs}$ *for all sufficiently large $(r,s) \in \mathbb{N}^2$.*

(ii) $\deg_{t_1}\Phi_{\eta|K}^{(V,W)} \le m = \mathrm{Card}\,\Delta$ *and* $\deg_{t_2}\Phi_{\eta|K}^{(V,W)} \le n = \mathrm{Card}\,\sigma$.

(iii) $\Phi_{\eta|K}^{(V,W)}$ *is an alternating sum of bivariate quasi-polynomials of the form $g(t_1)h(t_2)$ where $g(t_1)$ and $h(t_2)$ are (univariate) Ehrhart quasi-polynomials associated with rational conic polytopes.*

(iv) *The total degree and the coefficient of $t_l^m t_2^n$ of the quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1,t_2)$ are constants that do not depend on the set of difference-differential generators η of the extension L/K.*

This theorem generalizes the result on a bivariate difference-differential dimension polynomial proved in [4]. Furthermore, Theorem 3 allows one to assign a bivariate quasi-polynomial to a system of algebraic difference-differential (Δ-σ-) equations with weighted basic derivations and translations

$$f_i(y_1,\dots,y_q) = 0 \qquad (i = 1,\dots,p) \tag{1}$$

($f_i \in R = K\{y_1,\dots,y_q\}$ $(1 \le i \le p)$ where $K\{y_1,\dots,y_q\}$ denotes the ring of difference-differential polynomials in q variables over K) such that the Δ-σ-ideal P of R generated by the Δ-σ-polynomials $f_1,\dots,f_p$ is prime (e. g., to a system of linear difference-differential equations). Systems of this form arise in connection with systems of PDEs with weighted derivatives (see, for example, [7] and [8]) and their finite difference approximations.

In this case, the reflexive closure P^* of the Δ-σ-ideal P is also prime, so one can consider the quotient field of R/P^* as a finitely generated Δ-σ-field extension of K: $L = K\langle\eta_1,\dots,\eta_q\rangle$ where η_i is the canonical image of y_i in R/P^*. The corresponding bivariate dimension quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1,t_2)$ can be viewed as the Einstein's strength of the system (1) in the sense of the corresponding concepts for systems of partial differential and difference equations (see [6] and [5, Section 7.7] for detail descriptions of these concepts and their expressions as dimension polynomials).

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