Differential algebra with mathematical functions, symbolic powers, and anticommutative variables

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Computer algebra implementations of Differential Algebra typically require that the systems of equations to be tackled be rational in the independent and dependent variables and their partial derivatives. It is possible, however, to extend this computational domain and apply Differential Algebra techniques to systems of equations that involve arbitrary compositions of mathematical functions (elementary or special), fractional and symbolic powers, as well as anticommutative variables and functions. In this talk, this extension of the computational domain of Differential Algebra is explained, and examples of its implementation in the Maple computer algebra system, as well as of its use in the Maple ODE and PDE solvers, are shown.

The key observation regarding performing standard differential algebra operations on expressions that include mathematical functions is the fact that, but for rather few exceptions, they belong to a set of functions whose derivatives belong to the same set. For example, the derivative of a hypergeometric function is also a hypergeometric function, and with that the derivatives of all elementary and special functions that are particular cases of hypergeometric functions happen to belong to the same set as the functions themselves. It is then possible to represent each mathematical function of this group by an auxiliary function F_i that satisfies a differential equation, rational in the F_i, their derivatives and the independent variables (the mathematical functions' parameters). In brief, in the original system that includes mathematical functions, each of them is replaced by an auxiliary F_i, the differential equation it satisfies is added to the system, the differential algebra operations are performed, and at the end the F_i are substituted back by the mathematical functions they represent. As the simplest example of this, a system involving the exponential function of x is one where this function can be replaced by F and the equation $F' = F$ added to the system.

This rewriting of the original system by replacing mathematical functions by the F_i plus adding the rational differential equations they satisfy is called *rewriting the original system in differential polynomial form*, and the whole problem of performing differential algebra operations on systems that involve mathematical functions is thus reduced to this rewriting.

Returning to the representation problem, symbolic powers, say $F = x^n$, in turn satisfy $xF' - nF = 0$, and it is not difficult to see that the case where the mathematical function's arguments are not simple variables x_i, for example an exponential function of the form $e^{f(x)}$, can also be tackled as just described but for the addition of a change of variables to handle $f(x)$, provided that $f(x)$ itself can be written in differential polynomial form. In this way, for example, we find that $F = e^{x^n}$ satisfies $F''Fx - (F'x + F(n-1))F' = 0$.

This approach can be used as well for *derivatives evaluated at a point*, which appear frequently in the symbolic (exact) solution of systems of partial differential equations tackled using changes of variables, a standard operation in most methods, including the Lie symmetry and integrating factor approaches. Indeed, by differentiating one can see that, depending on the evaluation point, the derivative can be differentiated resulting again in closure (the objects and their derivatives happen to belong to the same set and therefore are suitable for a differential polynomial representation). For example, the function $F(x,t) = D(f)(x-t) + D(f)(x+t)$, where D is a differential operator and f is a mapping of one argument (a function of one variable), by means of this approach can be represented in differential polynomial form as $F_{xx} - F_{tt} = 0$. In the same way, one can represent integrals, provided that the integrand admits differential polynomial form; and in general, any arbitrary mathematical composition of operations (mathematical functions, symbolic powers, derivatives, integrals, etc.) with no restrictions to the levels of nesting, can be represented in differential polynomial form provided that the arguments of those operations in turn admit such rewriting.

Finally, the case of a PDE system involving anticommutative variables and functions can be reduced to the previous problem by expanding these functions in powers of the anticommutative variables. In view of the anticommutative character, these expansions truncate at first order in each anticommutative variable, so that each equation of the original system splits into equations without anticommutative variables, which can then be rewritten in differential polynomial form, tackled using differential algebra techniques, and at the end, the resulting equations can be recast as a system in the original anticommutative variables. As an example of an ODE involving anticommutative variables tackled using differential algebra techniques, consider Q as an anticommutative function, so that $Q^2 = 0$, then the ODE $Q'' - QQ' = 0$ has for solution $Q = (c_1 x + c_2)\lambda$, where λ is an anticommutative arbitrary constant. For a PDE example, consider an anticommutative function $Q(x,y,\theta)$ where Q and θ are anticommutative, then $Q_{x\theta} = 0$ has for solution $Q(x,y,\theta) = F_1(x,y)\lambda + F_2\theta$, where F_1 and F_2 are arbitrary commutative functions.

References

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