On finite difference approximations to the Korteveg-de Vries equation and its conservation laws

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Let ∂_x be the derivation operator w.r.t. x and $\mathscr{R} := \mathbb{Q}(a_1,\dots,a_i)\{u\}$ be the ordinary differential polynomial ring over the parametric field $\mathbb{Q}(a_1,\dots,a_i)$ of real constants. Based on the methodology of paper [1], we suggested in [2] an approach to algorithmic generation of finite difference approximations to the nonlinear evolution equations of the form

$$u_t = au_m + F(u_{m-1},\dots,u_1,u)\,, \quad 0 \neq a \in \mathbb{R}\,, \quad m \in \mathbb{N}_{>0}\,. \tag{1}$$

Here $u_k := \partial_x^k u$ $(0 \leq k \leq m)$, $u_0 := u$, $F \in \mathscr{R}$ is a differential polynomial of the order $m-1$ in ∂_x and such that there is a differential polynomial $P \in \mathscr{R}$ satisfying $F = \partial_x P$.

The class (1) contains the classical Korteveg-de Vries (KdV) equation which we shall write as

$$f = 0\,, \quad f := u_t + \alpha\, u\, u_x + \beta\, u_{xxx}\,, \quad u = u(t,x)\,, \quad \alpha, \beta \in \mathbb{R}\,. \tag{2}$$

The finite difference approximation (FDA) to Eq. (2), generated by the procedure described in [2] and based on application of difference Gröbner bases [3] reads

$$\begin{aligned} &\tilde{f} = 0\,, \quad \tilde{f} := \frac{u_j^{n+1} - u_j^n}{\tau} + \alpha \frac{(P_{j+1}^{n+1} - P_{j-1}^{n+1}) + (P_{j+1}^n - P_{j-1}^n)}{8h} \\ &+ \beta \frac{(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + (u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n)}{4h^3}\,. \end{aligned} \tag{3}$$

where $u_j^n := u(\tau \cdot n, h \cdot j)$ $(n, j \in \mathbb{Z})$ is the grid function which approximates $u(t,x)$ on the Cartesian solution grid with spacings $\tau := t_{n+1} - t_n, h := x_{j+1} - x_j$ and $P_j^n := (u^2)_j^n$, . The FDA (3) has accuracy $O(\tau^2, h^2)$ and is *consistent* with (2). Besides, as a difference scheme, it is implicit, and hence unconditionally *stable*. Therefore, the scheme (3) is *convergent*.

Apparently, the differential ideal $[\![f]\!]$, generated by f in (2), is radical, and the difference ideal $[\![\tilde{f}]\!]$, generated by $\tilde{f}$ in (3), is a perfect one (cf. [4]) in the inversive difference ring $\mathbb{Q}(\alpha,\beta)\{u\}$ with differences σ_t, $\sigma_x, \sigma_t^{-1}, \sigma_x^{-1}$ acting as

$$\sigma_t \circ u_j^n = u_j^{n+1}, \;\; \sigma_x \circ u_j^n = u_{j+1}^n, \;\; \sigma_t^{-1} \circ u_j^n = u_j^{n-1}, \;\; \sigma_x^{-1} \circ u_j^n = u_{j-1}^n\,.$$

Since $\tilde{f}$ is a Gröbner basis of $[\![\tilde{f}]\!]$, the consistency implies *s(strong)-consistency* [5] by the following theorem.

Theorem 1 [5] *A FDA $\tilde{F}$ to a (system of) differential polynomial(s) F is s-consistent iff every element in the Gröbner (standard) basis of the difference ideal generated by $\tilde{F}$ provides FDA to an element of the radical differential ideal generated by F.*

The property of s-consistency of $\tilde{f}$ with f means that any element in $[\![\tilde{f}]\!]$ is a FDA to an element in $[\![f]\!]$. Among elements in $[\![f]\!]$ there are infinitely many (local) *conservation laws*

$$\{\, \mathfrak{C}_i := \partial_t T_i + \partial_x X_i \in [\![f]\!] \Longrightarrow \frac{d}{dt}\int_{-\infty}^{\infty} T_i\, dx = -[X_i]_{-\infty}^{\infty} \mid i \in \mathbb{N}_{\geq 1}\,,\ T_i, X_i \in \mathscr{R}\}$$

where $T_i = T_i(u)$ are *densities* and $X_i = X_i(u)$ are *fluxes*.

The conservation laws of KdV admit algorithmic construction. There are computer algebra packages, e.g. the Maple package PDEBELLII [6], which recursively compute T_i and X_i . Then one can express $\mathfrak{C}_i$ via f with a help of the Maple package the DIFFERENTIALTHOMAS implementing differential Thomas decomposition [7]. The first five conservation laws presented in Table 1.

Table 1: Low order conservation laws of KdV in terms of f

i	$\mathfrak{C}_i$	$\mathrm{ord}_x(\mathfrak{C}_i)$
1	f	3
2	f_x	4
3	$f_{xx} + 2uf$	5
4	$f_{xxx} + uf_x^4 + u_x^4 f$	6
5	$f_{xxxx} + 6uf_{xx} + 5u_x f_x + 6u_{xx} f + 6u^2 f$	7
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Exact or approximate inheritance of conservation laws at the discrete level is one of the most important qualitative requirements to finite difference schemes [8]. Due to the s-consistency with (2), the discretization (3) approximately inherits all its conservation laws as the following theorem states. It is the main theoretical result of this note.

Theorem 2 *For each conservation law $\mathfrak{C}_i$ of KdV there is an element $\tilde{f}_i$ in the perfect difference ideal $[\![\tilde{f}]\!]$ such that $\tilde{f}_i$ approximates $\mathfrak{C}_i$ with the accuracy $O(\tau^2, h^2)$ corresponding to the accuracy of $\tilde{f}$.*

We illustrate this fact by the 3rd and 4th KdV conservation laws of Table 1. With regard to *forward and backward differences*

$$\Delta_p := \frac{1}{h}\,(\sigma_x - 1)\,, \qquad \Delta_m := \frac{1}{h}\,(1-\sigma_x^{-1})\,,$$

the spatial derivatives occurring in $\mathfrak{C}_3$ and $\mathfrak{C}_4$ are approximated, with the prescribed accuracy, by the elements in $\mathscr{R}$ and $[\![\tilde{f}]\!]$ as follows

$$\frac{1}{2}\,(\Delta_p+\Delta_m)\circ u \xrightarrow[h\longrightarrow 0]{} u_x + O(h^2)\,, \quad \frac{1}{2}\,(\Delta_p+\Delta_m)\circ \tilde{f} \xrightarrow[h\longrightarrow 0]{} f_x + O(h^2)\,,$$

$$\Delta_m\,\Delta_p\circ\tilde{f} \xrightarrow[h\longrightarrow 0]{} f_{xx}+O(h^2)\,,\;\; \Delta_p\,\Delta_m\,\Delta_p\circ\tilde{f} - \frac{h}{2}\,\Delta_m\,\Delta_p\,\Delta_m\,\Delta_p\circ\tilde{f} \xrightarrow[h\longrightarrow 0]{} f_{xxx}+O(h^2)\,.$$

We correlated numerical behavior of our scheme (3) with two other schemes taken from the book [9]. Both of them have the same accuracy $O(\tau^2,h^2)$ as (3).

Explicit Scheme I ([9], Eq.1.80)

$$u_i^{n+1} = u_i^{n-1} - \frac{\alpha\tau}{h}u_i^n\left(u_{i+1}^n - u_{i-1}^n\right) - \frac{\beta\tau}{h^3}\left(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n\right)\,.$$

stable for

$$\tau \le \frac{2h^3}{3\sqrt{3}\beta} \cong 0.384\frac{h^3}{\beta}\,.$$

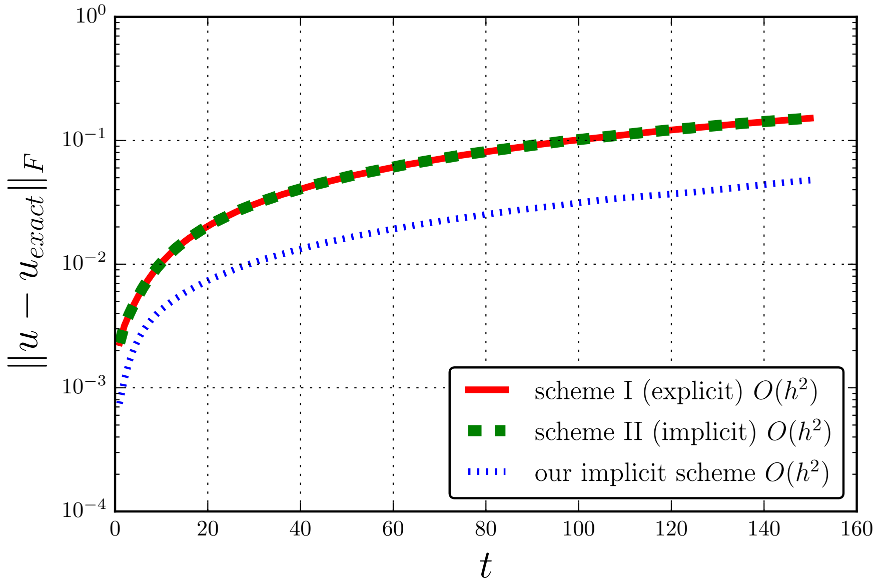
Implicit Scheme II ([9], Eq.1.96)

$$\frac{u_j^{n+1}-u_j^n}{\tau} + \frac{\alpha}{4h}\left[u_j^n\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right) + u_j^{n+1}\left(u_{j+1}^n - u_{j-1}^n\right)\right] +$$
$$+\frac{\beta}{4h^3}\left(\left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}\right) + \left(u_{j+2}^n - 2u_j^{n+1} + 2u_{j-1}^n - u_{j-2}^n\right)\right) = 0\,.$$

As a benchmark, we used the exact one-soliton solution

$$u_{\text{exact}}(x,y) = \frac{2k_1^2}{\cosh(k_1(x-4k_1^2t))^2}$$

to (2) with $\alpha=6$, $\beta=1$ and $k_1=0.4$. In so doing, we fixed $h=0.25$ and considered the solution in interval $-50\le x\le 50$ with periodic boundary conditions (cf. [9], p.49). The numerical inaccuracy was estimated by the Frobenius norm. The following picture shows numerical superiority of our scheme.



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