Higher-order symmetries and creation operators for linear equations via Maxima and SymPy

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Computation of symmetries of systems of partial differential equations is one of the oldest applications of computer algebra in the field of differential equations and mathematical physics. Already in the late eighties and early nineties several packages to compute symmetries have been developed in Macsyma, Reduce, Mathematica and Maple[1, 2]. Today, it is actually difficult to imagine *not* to use computer algebra when one faces analysis of complex differential (or difference) system. Skillful application of existent packages leads to efficient analysis of even very complicated systems like those encountered in theory of elasticity, see, e.g., [3].

Let us consider a system of partial differential equations:

$$U = 0, \tag{1}$$

and let X denotes the so-called inifinitesimal generator of symmetries which is a first-order linear partial differential operator. Then there exists the following infinitesimal criterion of symmetry (please see, e.g., [4]):

$$X^{(pr)}U|_{U=0} = 0, \tag{2}$$

where $X^{(pr)}$ is the prolongation of the operators X. The above formula has a simple geometric meaning: the symmetry of Q is such a tranformation (in the space of independent variables, dependent variables, and their derivatives) which leaves the hypersurface of solutions invariant. From the above condition a system of linear partial differential equations can be obtained to compute X. They are called *determining equations*. Even writing down all the determining equations is a very tedious procedure, ideally suited for computers.

In this contribution we, however, take advantage of the fact that for *linear* systems the way to obtain the determining equations is much simpler. Let us restrict ourselves to systems of the form:

$$Q\Psi = 0, \tag{3}$$

where Q is a (variable-coefficient) matrix partial differential operator, and Ψ is a vector of dependent variables. Then, a first-order matrix linear partial differential operator L is called a symmetry operator if and only if [5]:

$$[L, Q] - RQ = 0, \tag{4}$$

where $[,]$ denotes the commutator and R is a function of independent variables.

An important point is that the Eq. (4) can easily be generalized to the higher-order symmetry operators [5]. For instance, in the second-order case we have the following condition:

$$[L^{(2)}, Q] - R^{(1)}Q = 0, \tag{5}$$

where $L^{(2)}$ is a second-order, and $R^{(1)}$ - a first-order linear partial differential operators. Unlike the operators L, operators $L^{(2)}$ which satisfy (5) usually do not form a Lie algebra. Computing operators $L, L^{(2)}$ from Eqs. (4, 5) is by far simpler that from (2) but still sufficiently difficult as to require assistance from the computer algebra systems.

We have, in particular, applied both Maxima and SymPy to study the following Schrödinger equation:

$$\left(i\frac{\partial}{\partial t} - H\right)\Psi = 0, \tag{6}$$

where t denotes time and H - a Hamiltonian operator which is given in the representation of second-quantization as:

$$H = \sum_j \alpha_j a_j + \sum_{j,k} \beta_{j,k} a_j^\dagger a_k + \sum_{j,k,l,m} \gamma_{j,k,l,m} a_j^\dagger a_k^\dagger a_l a_m + h.c.,$$

where a_j, $a_k^\dagger$ are the annihilation and creation operators which satisfy:

$$[a_j, a_k^\dagger] = \delta_{jk}, \tag{7}$$

δ_{jk} is the Kronecker delta, "h.c." denotes Hermitian conjugate symbol while α_j, $\beta_{j,k}$, and $\gamma_{j,k,l,m}$ are complex constants. To apply computer algebra, we could, in principle, work directly with the above Hamiltonian using only (7). However, we have found it convenient to use the following Bargmann representation:

$$a_j \to \frac{\partial}{\partial z_j} \quad \text{and} \quad a_k^\dagger \to z_k.$$

In this representation, Ψ becomes a function of time and an analytic function of all z_j.

Using (independently) Maxima and SymPy we have determined the first-, second- and third-order symmetries for a generalized Bose-Hubbard model which describes systems of interacting bosons on a lattice. We also found first- and second-order generalized creation ($A^\dagger$) and annihilation A operators for such model; they have to satisfy the relations:

$$[H,A^\dagger] = A^\dagger \quad \text{and} \quad [H,A] = -A.$$

We have found it expedient to work with Maxima and SymPy firstly in the interactive modes, and write the corresponding scripts only later. Regarding Maxima, we observe that its feature which allows to use functions as first-order variables, inherited from Lisp, is a particular advantage. In several cases the symmetries obtained could be used to provide us with separation of variables. In other cases, special interesting exact solutions have been found.

References

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