Generalized Weyl algebras and diskew polynomial rings

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*talk-GWA-diskew.tex

Generalized Weyl algebras $D(\sigma, a)$ with central element a

Definition. Let D be a ring, $\sigma \in$ Aut(D), $a \in Z(A)$. The **generalized Weyl algebra** of rank 1 (GWA) $D(\sigma, a) = D[x, y; \sigma, a]$ is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \text{ and } yd = \sigma^{-1}(d)y \text{ for all } d \in D,$$

$$yx = a \text{ and } xy = \sigma(a).$$

Many popular algebras of small Gelfand-Kirillov dimension are GWAs: the first Weyl algebra A_1 and its quantum analogue, the *quantum plane*, the *quantum sphere*, $Usl(2)$, $U_q sl(2)$, the *Heizenberg* algebra and its quantum analogues, the 2×2 quantum matrices, the *Witten's* and *Woronowic's* deformations, Noetherian down-up algebras, etc.

Examples of generalized Weyl algebras where a is central.

1. The (first) **Weyl algebra** $A_1 = K\langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ is the GWA

$$A_1 = K[h][x, y := \partial; \sigma, a = h], \ \ \sigma(h) = h - 1.$$

2. The **quantum plane** $\Lambda = K\langle x, y \,|\, xy = qyx\rangle$ where q is a central unit of K is the GWA

$$\Lambda = K[h][x, y; \sigma, a = h], \ \ \sigma(h) = qh.$$

3. For q, $h = q - q^{-1} \in K = \mathbb{C}$, the algebra $U_q = U_q sl(2)$ is generated by X, Y, H_- and H_+ subject to the defining relations:

$$H_+H_- = H_-H_+ = 1, \ \ XH_\pm = q^{\pm 1}H_\pm X,$$

$$YH_\pm = q^{\mp 1}H_\pm Y, \ \ [X, Y] = \frac{H_+^2 - H_-^2}{h}.$$

The algebra U_q is a GWA,

$$U_q \simeq K[C, H, H^{-1}](\sigma, a),$$

$$a = C + \Big(H^2/(q^2-1) - H^{-2}/(q^{-2}-1)\Big)/2h$$

where $\sigma(H) = qH,\ \sigma(C) = C$.

4. **Woronowicz's deformation** $V = K\langle V_0, V_-, V_+\rangle$:

$s^2V_0V_+ - s^{-2}V_+V_0 = V_+,\ s^2V_0V_- - s^{-2}V_-V_0 = -V_-,$

$$s^{-1}V_+V_- - sV_-V_+ = V_0.$$

The algebra V is a GWA,

$$V \simeq K[u,v](\sigma, a = v)$$

where $\sigma : u \to s^2(s^2u - 1),\ v \to s^2v + su$.

5. **Witten's first deformation** $E = K\langle E_0, E_-, E_+\rangle$:

$$[E_0, E_+]_p := pE_0E_+ - p^{-1}E_+E_0 = E_+,$$

$$[E_-, E_0]_p = E_-,$$

$$[E_+, E_-] = E_0 - (p - 1/p)E_0^2,$$

where $p \neq 0, \pm 1, \pm i \in K$. The element

$$C = E_- E_+ + \frac{E_0(E_0 + p)}{p(p^2 + 1)}$$

is central in E. Witten's first deformation is a GWA,

$$E \simeq K[C, H](\sigma, a = C - H(H+1)/(p + p^{-1}))$$

where $\sigma : C \to C$, $H \to p^2(H - 1)$.

6. **The quantum group**

$\mathcal{O}_{q^2}(\mathrm{so}(K, 3)) \simeq K[H, C](\sigma, a = C + H^2/q(1+q^2))$,

$\sigma(H) = q^2 H$ and $\sigma(C) = C$.

Generalized Weyl algebras with two endomorphisms and a left normal element a

The elements a and $\sigma(a)$ are left normal in D. An element d of a ring D is called **left normal** (resp., **normal**) if $dD \subseteq Dd$ (resp., $Dd = dD$).

Definition. Let D be a ring, σ and τ be ring endomorphisms of D and $a \in D$ be s. t.

$\tau\sigma(a) = a$, $ad = \tau\sigma(d)a$ and $\sigma(a)d = \sigma\tau(d)\sigma(a)$

for all $d \in D$. The **generalized Weyl algebra** (GWA) of rank 1,

$$A = D(\sigma, \tau, a) = D[x, y; \sigma, \tau, a],$$

is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \text{ and } yd = \tau(d)y \text{ for all } d \in D,$$

$$yx = a \text{ and } xy = \sigma(a).$$

Theorem 1 *The GWA $A = D[x, y; \sigma, \tau, a]$ is a $\mathbb{Z}$-graded ring $A = \oplus_{i \in \mathbb{Z}} A_i$ where $A_i = Dv_i \simeq {}_D D$, $v_0 = 1$, $v_i = x^i$ and $v_{-i} = y^i$ for $i \geq 1$. In particular, the module ${}_D A$ is free.*

Simplicity criteria for generalized Weyl algebras. Let D be a ring and σ be its ring endomorphism. An ideal I of D is called σ-**stable** if $\sigma(I) \subseteq I$. The ring D is called a σ-**simple** ring iff 0 and D are the only σ-stable ideals of the ring D. An endomorphism σ is **inner** if $\sigma = \omega_u$ for some unit $u \in D$ ($\sigma(d) = udu^{-1}$ for all $d \in D$). Then $\sigma \in \mathrm{Aut}(D)$.

Theorem 2 *Let $A = D[x, y; \sigma, \tau, a]$ be a GWA such that the elements a and $\sigma(a)$ are right normal in D. Then the following statement are equivalent.*

1. *A is a simple ring.*

2.(a) The elements a and $\sigma(a)$ are regular in D,

(b) D is a σ-simple ring,

(c) for all $i \geq 1$, σ^i is not an inner automorphism of the ring D, and

(d) for all $i \geq 1$, $Da + D\sigma^i(a) = D$.

3.(a) The elements a and $\sigma(a)$ are regular in D,

(b) D is a τ-simple ring,

(c) for all $i \geq 1$, τ^i is not an inner automorphism of the ring D, and

(d) for all $i \geq 1$, $D\sigma(a) + D\tau^i\sigma(a) = D$.

If one of the equivalent conditions holds then σ and τ are automorphisms of D.

The centre of a GWA. The next theorem describes the centre of a GWA.

Theorem 3 *Let* $A = D[x, y; \sigma, \tau, a]$*. Then the centre of* A*,* $Z(A) = \bigoplus_{i \in \mathbb{Z}} D_i v_i$*, is a* $\mathbb{Z}$*-graded subring of* A *where*

$$D_0 = Z(D)^{\sigma,\tau} := \{d \in D \,|\, \sigma(d) = d, \tau(d) = d\}$$

and, for $i \geq 1$*,* $D_i = \{\alpha \in D^{\sigma} \,|\, d\alpha = \alpha\sigma^i(d)$ *for all* $d \in D\}$ *and* $D_{-i} = \{\beta \in D^{\tau} \,|\, d\beta = \beta\tau^i(d)\beta$ *for all* $d \in D\}$*.*

Involutions on GWAs. An anti-isomorphism $*$ of a ring R ($(ab)^* = b^*a^*$ for all $a, b \in R$) is called an **involution** if $a^{**} = a$ for all elements $a \in R$.

The Weyl algebra A_1 admits the canonical involution $*$:

$$x^* = \partial \text{ and } \partial^* = x.$$

The Weyl algebra A_1 is the GWA

$$K[h][x, \partial; \sigma, \sigma^{-1}, a = h]$$

and the involution $*$ respects the subalgebra $K[h]$: $K[h]^* = K[h]$ since $h^* = h$. So, the involution $*$ on A_1 can be seen as an extension of the (trivial) involution on the commutative algebra $K[h]$ to A_1.

Lemma 4 *Let $A = D[x, y; \sigma, \tau, a]$ be a GWA. Suppose that $*$ is an involution of the ring D such that $\sigma * \tau = *$, $\tau * \sigma = *$, $a^* = a$ and $\sigma(a)^* = \sigma(a)$. Then the involution $*$ can be extended to an involution $*$ of A by the rule $x^* = y$ and $y^* = x$.*

Corollary 5 *Let $A = D[x, y; \sigma, \tau, a]$ be a GWA where D is a commutative ring, σ and τ are automorphisms of D such that $\tau = \sigma^{-1}$. Then the trivial involution on D can be extended to an involution $*$ of D by the rule $x^* = y$ and $y^* = x$.*

Diskew polynomial rings

Definition. Let D be a ring, σ and τ be its ring endomorphisms, ρ and b be elements of D such that, for all $d \in D$,

$$\sigma\tau(d)\rho = \rho\tau\sigma(d) \text{ and } \sigma\tau(d)b = bd, \qquad (1)$$

The **diskew polynomial ring** (DPR)

$$E := D(\sigma, \tau, b, \rho) := D[x, y; \sigma, \tau, b, \rho]$$

is a ring generated by D, x and y subject to the defining relations:

$$xd = \sigma(d)x \text{ and } yd = \tau(d)y \text{ for all } d \in D,$$

$$xy - \rho yx = b.$$

By (1), b is a left normal element of D.

Example. The quantum plane $\Lambda = K\langle p, q \mid pq = \lambda qp\rangle$ (over a field K where $\lambda \in K^*$) is a skew polynomial ring $\Lambda = K[q][p; \nu]$ where $\nu(q) = \lambda q$. Then the algebra

$$E = \Lambda[x, y; \nu^\alpha, \nu^\beta, \eta p^{\alpha+\beta}, \rho]$$

is a diskew polynomial ring where $\eta, \rho \in K^*$ and $\alpha, \beta \in \mathbb{N}$.

The diskew polynomial rings are a generalization of the following class of rings which is a part of the class of diskew polynomial rings.

Definition. Let D be an ring, $\sigma \in \mathrm{Aut}(D)$, $b, \rho \in Z(D)$ and ρ is invertible and $\sigma(\rho) = \rho$. Then $E := D\langle \sigma; b, \rho\rangle := D\langle X, Y; \sigma, b, \rho\rangle$ is a ring generated by D, X and Y subject to the defining relations: For all $\alpha \in D$,

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y,$$

$$XY - \rho YX = b.$$

Clearly, $E = D[X, Y; \sigma, \sigma^{-1}, b, \rho]$ is a diskew polynomial ring.

Example. $Usl(2) = K\langle X, H, Y \mid [H, X] = X, [H, Y] = -Y, [X, Y] = 2H\rangle$.

$$Usl(2) = K[H]\langle X, Y; \sigma, 2H, 1\rangle, \ \ \sigma(H) = H - 1.$$

In 90s, there were many examples like this, various 'quantum deformations' of $Usl(2)$, with a ring D which is a 'small' commutative ring, eg, $U_q(sl_2)$, $\mathcal{O}_{q^2}(\text{so}(K,3))$, the quantum Weyl algebra, the quantum plane, etc.

Diskew polynomial rings are GWAs when ρ is a unit. If the element ρ is a unit in D then every diskew polynomial ring is a generalized Weyl algebra.

Theorem 6 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring. Suppose that ρ is a unit in D. Then x and y are left regular elements of E and the ring $E = \mathcal{D}[x, y; \sigma, \tau, a = h]$ is a GWA with base ring $\mathcal{D} := D[h; \tau\sigma]$ which is a skew polynomial ring, σ and τ are ring endomorphisms of $\mathcal{D}$ that are extensions of the ring endomorphisms σ and τ of D, respectively, defined by the rule $\sigma(h) = \rho h + b$ and $\tau(h) = \tau(\rho^{-1})(h - \tau(b))$.*

Simplicity criterion for diskew polynomial rings when ρ is a unit.

Theorem 7 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit in D and $\nu := \tau\sigma$ is an epimorphism. The following statements are equivalent.*

1. *The ring E is a simple ring.*

2.(a) *The endomorphisms σ and τ of D are automorphisms,*

(b) *the ring D is a σ-simple ring,*

(c) *for each natural number $n \geq 1$ there is no element $p = h^n + \sum_{i=0}^{n-1} d_i h^i \in \mathcal{D}$, where $d_i \in D$, such that*

   i. *for all elements $d \in D$, $pd = \nu^n(d)p$, i.e., $d_i d = \nu^{n-i}(d) d_i$ for $i = 0, 1, \ldots, n-1$,*

ii. $\sigma(p) = \rho_n^\nu p$ where $\rho_n^\nu = \rho\nu(\rho)\cdots\nu^{n-1}(\rho)$, and

iii. $[h, p] = 0$, i.e, $\nu(d_i) = d_i$ for $i = 0, 1, \ldots, n-1$, and

(d) the elements $b_i \in D$ (see (**??**)), where $i \geq 1$, are units in D. In particular, $b = b_1 \in D$ is a unit.

3.(a) The endomorphisms σ and τ of D are automorphisms,

(b) the ring D is a τ-simple ring,

(c) for each number $n \geq 1$ there is no element $p' = h'^n + \sum_{i=0}^{n-1} d_i' h'^i \in \mathcal{D} = D[h', \mu := \sigma\tau]$, where $d_i' \in D$ and $h' = \sigma(h)$, such that

i. for all elements $d \in D$, $p'd = \mu^n(d)p'$, i.e., $d_i'd = \mu^{n-i}(d)d_i'$ for $i = 0, 1, \ldots, n-1$, *ii.* $\tau(p') = (\rho^{-1})^{\mu}_{n} p$ *where*

$(\rho^{-1})^{\mu}_{n} := \rho^{-1}\mu(\rho^{-1})\cdots\mu^{n-1}(\rho^{-1})$, *and*

iii. $[h', p'] = 0$, *i.e,* $\mu(d'_i) = d_i$ *for* $i = 0, 1, \dots, n-1$, *and*

(d) the elements $b'_i \in D$, *where* $i \geq 1$, *are units in* D. *In particular,* $b = b_1 \in D$ *is a unit.*

Every simple ring is necessarily an algebra. Theorem 8 and Theorem 9 are refined versions of Theorem 7 in zero and prime characteristic, respectively.

Simplicity criterion for DPRs in characteristic zero. If the ring D is a $\mathbb{Q}$-algebra the condition (c) in Theorem 7 can be simplified.

Theorem 8 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit in D, $\nu = \tau\sigma$ is an epimorphism and D is a $\mathbb{Q}$-algebra. The following statements are equivalent.*

1. *The ring E is a simple ring.*

2.(a) *The endomorphisms σ and τ of D are automorphisms,*

(b) *the ring D is a σ-simple ring,*

(c) *there is no element $\alpha \in D$ such that $\rho\alpha - \sigma(\alpha) = b$ and $\alpha d = \nu(d)\alpha$ for all elements $d \in D$.*

(d) *the elements $b_i \in D$, where $i \geq 1$, are units in D. In particular, $b = b_1 \in D$ is a unit.*

Simplicity criterion for DPRs in prime characteristic p. If the ring D is a $\mathbb{F}_p$-algebra the condition (c) in Theorem 7 can be replaced by more explicit conditions (where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$).

Theorem 9 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit in D, $\nu = \tau\sigma$ is an epimorphism and D is a $\mathbb{F}_p$-algebra. The following statements are equivalent.*

1. *The ring E is a simple ring.*

2.(a) *The endomorphisms σ and τ of D are automorphisms,*

(b) *the ring D is a σ-simple ring,*

(c) *for each natural number $n \geq 0$ there is no element $p' = h^{p^n} + \sum_{i=0}^{n-1} \alpha_i h^{p^i} + \alpha$, where $\alpha, \alpha_i \in D$, such that*

i. for all $d \in D$, $pd = \nu^{p^n}(d)p$, i.e. $\alpha d = \nu^{p^n}(d)\alpha$ and $\alpha_i d = \nu^{p^n - p^i}(d)\alpha_i$ for $i = 0, 1, \ldots, n-1$,

ii. $\sigma(p') = \rho^{\nu}_{p^n} p'$,

iii. $[h, p'] = 0$, i.e. $\nu(\alpha) = \alpha$ and $\nu(\alpha_i) = \alpha_i$ for $i = 0, 1, \ldots, n-1$.

(d) the elements $b_i \in D$, where $i \geq 1$, are units in D. In particular, $b = b_1 \in D$ is a unit.

3.(a) The endomorphisms σ and τ of D are automorphisms,

(b) the ring D is a σ-simple ring,

(c) there is no element $\alpha \in D$ such that $\rho\alpha - \sigma(\alpha) = b$, $\nu(\alpha) = \alpha$ and $\alpha d = \nu(d)\alpha$ for all elements $d \in D$, and for each natural number $n \geq 1$ there are no elements $\alpha, \alpha_0, \ldots, \alpha_n$ such that

i. *for all* $d \in D$, $\alpha d = \nu^{p^n}(d)\alpha$ *and* $\alpha_i d = \nu^{p^n - p^i}(d)\alpha_i$ *for* $i = 0, 1, \ldots, n-1$,

ii. $\sigma(\alpha_i) = \rho^{\nu}_{p^n - p^i}\alpha_i$ *for* $i = 0, 1, \ldots, n-1$, *and* $\rho^{\nu}_{p^n}\alpha - \sigma(\alpha) = b^{p^n} + \sum_{i=0}^{n-1} \sigma(\alpha_i) b^{p^i}$,

iii. $\nu(\alpha) = \alpha$ *and* $\nu(\alpha_i) = \alpha_i$ *for* $i = 0, 1, \ldots, n-1$.

(d) *the elements* $b_i \in D$, *where* $i \geq 1$, *are units in* D. *In particular,* $b = b_1 \in D$ *is a unit.*

The canonical left normal element C of a diskew polynomial ring. Theorem 10 is a criterion for an element $C = h + \alpha$ (where $\alpha \in D$) to be a left normal element in E, it is a key moment in the proof of Theorem 8 and Theorem 9 (together with "Meeting the p-neighbour method').

Theorem 10 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit, $\mathcal{D} = D[h; \nu = \tau\sigma]$ and $C = h + \alpha$ where $h = yx$ and $\alpha \in D$. The following statements are equivalent.*

1. *The element C is left normal in E.*

2. *$\rho\alpha - \sigma(\alpha) = b$, $\nu(\alpha) = \alpha$ and $\alpha d = \nu(d)\alpha$ for all elements $d \in D$.*

   If one of the equivalent conditions holds then $[h, C] = 0$ and

(a) $C = \rho^{-1}(xy + \sigma(\alpha))$, $xC = \rho Cx$ and $yC = \tau(\rho^{-1})Cy$.

(b) $E \simeq D[C;\nu][x,y;\sigma,\tau,a := C - \alpha]$ is a GWA where $\sigma(C) = \rho C$ and $\tau(C) = \tau(\rho^{-1})C$.

(c) The element C is left normal, left regular in E and $E/(C) \simeq D[x,y;\sigma,\tau,-\alpha]$ is a GWA.

(d) The element C is a normal element in E iff $\mathrm{im}(\nu) = D$.

(e) The element C is regular iff C is right regular iff $\mathrm{ker}(\nu) = 0$.

(f) The element C is a normal, regular element iff ν is an automorphism of D.

The canonical central element C of a diskew polynomial ring (under certain conditions). The next corollary is a criterion for an element $C+\alpha$ (where $\alpha \in D$) to be a central element in E.

Corollary 11 *Let $E = D[x, y; \sigma, \tau, b, \rho]$ be a diskew polynomial ring such that ρ is a unit, $\mathcal{D} = D[h; \nu = \tau\sigma]$ and $C = h + \alpha$ where $h = yx$ and $\alpha \in D$. The following statements are equivalent.*

1. *The element C is a central element of E.*

2. *$\rho = 1$, $\nu = 1$, $\alpha - \sigma(\alpha) = b$, and the element α belongs to the centre of D.*

   If one of the equivalent conditions holds then

   (a) $C = xy + \sigma(\alpha)$.

(b) $E \simeq D[C][x, y; \sigma, \tau, a := C - \alpha]$ *is a GWA where* $\sigma(C) = C$ *and* $\tau(C) = C$.

(c) The element C *is a regular element of* E.

The centre of a DPR. The next theorem describes the centre of a DPR.

Theorem 12 *Let E be a diskew polynomial ring such that $\rho = 1$ is a unit and C is a central element of E, that is $E = D[C][x, y; \sigma, \tau, a = C - \alpha]$ is a GWA where $\sigma(C) = C$, $\tau(C) = C$, $\tau\sigma = \mathrm{id}_D$, $\alpha - \sigma(\alpha) = b$ and $\alpha \in Z(D)$ (see Corollary 11). Then the centre of E, $Z(E) = \mathcal{Z}[C]$, is a polynomial ring in C over a ring $\mathcal{Z} = \oplus_{i\in\mathbb{Z}} D_i v_i$ where $D_0 = Z(D)^{\sigma,\tau}$ and, for $i \geq 1$, $D_i = \{\alpha \in D^{\sigma} \,|\, d\alpha = \alpha\sigma^i(d)$ for all $d \in D\}$ and $D_{-i} = \{\beta \in D^{\tau} \,|\, d\beta = \beta\tau^i(d)\beta$ for all $d \in D\}$.*

Rings with enough normal elements. We say that a ring has **enough normal elements** if each nonzero ideal contains a normal element. All commutative rings have enough normal elements. In a similar way, a ring that has **enough left/right normal elements** is defined. The next corollary provides examples of DPRs/GWAs that have enough regular normal elements.

Corollary 13 *Let* $E = D[x, y; \sigma, \tau, b, \rho]$ *be a diskew polynomial ring such that* ρ *is a unit in* D *and* $\nu = \tau\sigma$ *is an epimorphism. Suppose that the ring* D *is* σ*-simple (resp.,* τ*-simple);* σ *(resp.,* τ*) is an automorphism of* D *and the elements* b_i *(resp.,* b'_i*) are units in* D *for all* $i \geq 1$*. Then the ring* E *has enough regular normal elements.*

Generalized Weyl algebras of rank n

Classical generalized Weyl algebras. Let D be a ring, $\sigma = (\sigma_1, \dots, \sigma_n)$ an n-tuple of commuting automorphisms of D, $a = (a_1, \dots, a_n)$ an n-tuple of elements of $Z(D)$ s. t. $\sigma_i(a_j) = a_j$ for all $i \neq j$. The **(classical) generalized Weyl algebra**

$$A = D(\sigma, a) = D[x, y; \sigma, a]$$

of rank n is a ring generated by D and $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the defining relations:

$$y_i x_i = a_i, \;\; x_i y_i = \sigma_i(a_i), \;\; x_i d = \sigma_i(d) x_i, \;\; \text{and}$$

$$y_i d = \sigma_i^{-1}(d) y_i \;\; \text{for all} \;\; d \in D,$$

$$[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 0, \;\; \text{for all} \;\; i \neq j,$$

where $[x, y] = xy - yx$.

Example. The n'th *Weyl algebra*, $A_n = A_n(K) = K\langle x_1, ..., x_n, y_1, ..., y_n\rangle$:

$[y_i, x_i] = \delta_{ij}$ and $[x_i, x_j] = [y_i, y_j] = 0$ for all i, j,

where δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a GWA $A = D[x, y; \sigma; a]$ of rank n where $D = K[H_1, ..., H_n]$, $\sigma = (\sigma_1, \dots, \sigma_n)$ where $\sigma_i(H_j) = H_j - \delta_{ij}$ and $a = (H_1, \dots, H_n)$. The map

$$A_n \to A, \quad x_i \mapsto x_i, \quad y_i \mapsto y_i, \quad i = 1, \dots, n,$$

is an algebra isomorphism (notice that $y_i x_i \mapsto H_i$).

Iterated generalized Weyl algebras.

Proposition 14 *Let*

$$A = D[x_1, y_1; \sigma_1, \tau_1, a_1] \dots [x_n, y_n; \sigma_n, \tau_n, a_n]$$

be an iterated GWA of rank n. Then $A = \oplus_{\alpha \in \mathbb{Z}^n} D v_\alpha$ is a direct sum of the free left D-modules ${}_D D v_\alpha \simeq D$ where for $\alpha = (\alpha_1, \dots, \alpha_n)$, $v_\alpha = v_{\alpha_1}(1) \cdots v_{\alpha_n}(n)$ and

$$v_{\alpha_i}(i) = \begin{cases} x_i^{\alpha_i} & \text{if } \alpha_i \geq 0, \\ y_i^{-\alpha_i} & \text{if } \alpha_i < 0. \end{cases}$$

Generalized Weyl algebras. Let A be a ring and σ its endomorphism. A subring B of A is called σ-**invariant** if $\sigma(B) \subseteq B$.

Definition. An iterated generalized Weyl algebra

$$A = D[x_1, y_1; \sigma_1, \tau_1, a_1] \dots [x_n, y_n; \sigma_n, \tau_n, a_n]$$

is called a **generalized Weyl algebra** of rank n if $a_1, \dots, a_n \in D$, the ring D is σ_i- and τ_i-invariant for all $i = 1, \dots, n$; and for all integers $i, j = 1, \dots, n$ such that $i > j$:

$$\sigma_i(x_j) = \lambda_{ij} x_j, \ \ \sigma_i(y_j) = \lambda'_{ij} y_j,$$

$$\tau_i(x_j) = \mu_{ij} x_j, \ \ \tau_i(y_j) = \mu'_{ij} y_j,$$

for some elements $\lambda_{ij}, \lambda'_{ij}, \mu_{ij}$ and μ'_{ij} of the ring D. The elements $\Lambda = (\lambda_{ij})$, $\Lambda' = (\lambda'_{ij})$, $M = (\mu_{ij})$ and $M' = (\mu'_{ij})$ are called the **defining coefficients** of A. The n-tuples of endomorphisms $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ are called the **defining endomorphisms** of A, and the n-tuple of elements $a = (a_1, \dots, a_n)$ is called the **defining elements** of A. The GWA A of rank n is denoted by

$$A = D[x, y; \sigma, \tau, \Lambda, \Lambda', M, M']$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

An element $\Lambda = (\lambda_{ij})$ (where $1 \leq j < i \leq n$) is called a **lower triangular half-matrix** with coefficients in D. The set of all such elements is denoted by $L_n(D)$. The next proposition describes GWAs of rank n via generators and defining relations.

Proposition 15 *Let D be a ring, $\sigma = (\sigma_i)$ and $\tau = (\tau_i)$ be n-tuples of ring endomorphisms of D, $a = (a_i) \in D^n$, and $\Lambda = (\lambda_{ij}), \Lambda' = (\lambda'_{ij}), M = (\mu_{ij}), M' = (\mu'_{ij}) \in L_n(D)$ be such that the following conditions hold: For all $i = 1, \ldots, n$ and $d \in D$,*

$$\tau_i\sigma_i(a_i) = a_i, \;\; a_i d = \tau_i\sigma_i(d)a_i \text{ and}$$

$$\sigma_i(a_i)d = \sigma_i\tau_i(d)\sigma_i(a_i);$$

for all $i > j$,

$$a_i = \tau_i(\lambda_{ij})\mu_{ij}\sigma_j(a_i) = \tau_i(\lambda'_{ij})\mu'_{ij}\tau_j(a_i),$$

$$\sigma_i(a_i) = \sigma_i(\mu_{ij})\lambda_{ij}\sigma_j\sigma_i(a_i) = \sigma_i(\mu'_{ij})\lambda'_{ij}\tau_j\sigma_i(a_i);$$

...

The GWA of rank n, $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']$, *is a ring generated by* D, $x_1, \dots, x_n$ *and* $y_1, \dots, y_n$ *subject to the defining relations: For all* $i = 1, \dots, n$ *and* $d \in D$,

$$x_i d = \sigma_i(d) x_i, \;\; y_i d = \tau_i(d) y_i, \;\; y_i x_i = a_i \;\text{ and}$$

$$x_i y_i = \sigma_i(a_i);$$

for all $i > j$,

$$x_i x_j = \lambda_{ij} x_j x_i, \;\; x_i y_j = \lambda'_{ij} y_j x_i, \;\; y_i x_j = \mu_{ij} x_j y_i$$

$$\text{and} \;\; y_i y_j = \mu'_{ij} y_j y_i.$$

Example. If $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathrm{Aut}(D)^n$ is an n-tuple of *commuting* automorphisms of the ring D, $\tau := \sigma^{-1} = (\sigma_1^{-1}, \dots, \sigma_n^{-1})$, $a = (a_1, \dots, a_n) \in Z(D)$ and $\sigma_i(a_j) = a_j$ for all $i \neq j$; and $\lambda_{ij} = \lambda'_{ij} = \mu_{ij} = \mu'_{ij} = 1$ for all $i > j$, then the GWA A of rank n is a *classical* GWA of rank n, that is $A = D[x, y; \sigma, a]$.

Each normal, regular element α of a ring D determines the automorphism ω_α of D by the rule $\alpha d = \omega_\alpha(d)\alpha$ for all $d \in D$. The next proposition gives plenty of examples of GWAs of rank n.

Proposition 16 *Let D be a ring, $\theta_1, \ldots, \theta_n$ commuting automorphisms of the ring D, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ regular, normal elements of D. Then $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']$ be a GWA of rank n where*

$$\begin{aligned}
\sigma_i &= \theta_i \omega_{\beta_i}, \;\; \tau_i = \omega_{\alpha_i} \theta_i^{-1}, \;\; a_i = \alpha_i \beta_i, \\
\lambda_{ij} &= \theta_i(\beta_i) \theta_i \theta_j (\beta_j \beta_i^{-1}) \theta_j (\beta_j^{-1}), \\
\lambda'_{ij} &= \theta_i(\beta_i \alpha_j) \cdot \theta_j^{-1} \theta_i (\beta_i^{-1}) \alpha_j^{-1}, \\
\mu_{ij} &= \alpha_i \theta_i^{-1} \theta_j (\beta_j) \theta_j (\alpha_i^{-1} \beta_j^{-1}), \\
\mu'_{ij} &= \alpha_i \theta_i^{-1} (\alpha_j) \theta_j^{-1} (\alpha_i^{-1}) \alpha_j^{-1},
\end{aligned}$$

provided $\lambda_{ij}, \lambda'_{ij}, \mu_{ij}, \mu_{ij} \in D$.

A $\mathbb{Z}^n$-grading of a GWA of rank n. Every GWA of rank n, $A = D[x, y; \sigma, \tau, a, \Lambda, \Lambda', M, M']$, is a $\mathbb{Z}^n$-graded algebra

$$A = \oplus_{\alpha \in \mathbb{Z}^n} D v_\alpha$$

where for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$,

$v_\alpha = v_{\alpha_1}(1) v_{\alpha_2}(2) \cdots v_{\alpha_n}(n)$ and

$$v_{\alpha_i}(i) := \begin{cases} x_i^{\alpha_i} & \text{if } \alpha_i \geq 0, \\ y_i^{-\alpha_i} & \text{if } \alpha_i < 0. \end{cases}$$

Notice that the order in the product for v_α is important and, in general, cannot be changed. Moreover, the left D-module $D v_\alpha$ is free of rank 1. For all elements $\alpha, \beta \in \mathbb{Z}^n$,

$$v_\alpha v_\beta = (\alpha, \beta) v_{\alpha+\beta}$$

for some (explicit) elements $(\alpha, \beta) \in D$. For all elements $\alpha \in \mathbb{Z}^n$ and $d \in D$, $v_\alpha d = \sigma^\alpha(d) v_\alpha$ where $\sigma^\alpha := \sigma(1, \alpha_1) \cdots \sigma(n, \alpha_n)$ and

$$\sigma(i, \alpha_i) := \begin{cases} \sigma_i^{\alpha_i} & \text{if } \alpha_i \geq 0, \\ \tau_i^{-\alpha_i} & \text{if } \alpha_i < 0. \end{cases}$$