Towards a symbolic package for systems of nonlinear difference equations

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Outline

- Thomas decomposition for differential systems
- Difference algebra
- Towards a symbolic package for nonlinear difference equations

1. Thomas decomposition for differential systems

Systems of linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm computes a vector space basis for power series solutions

(Maurice Janet, $\sim$ 1920)

Janet division

The possible ways of decomposing $\mathrm{Mon}(x_1, \ldots, x_n)$-multiple closed sets into disjoint cones are studied as

involutive divisions (Gerdt, Blinkov et. al.)

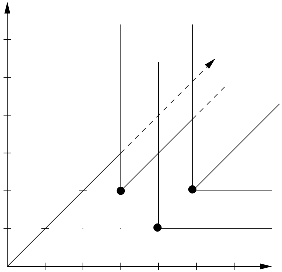
<u>Janet division:</u>

Let $G \subset \mathrm{Mon}(x_1, \ldots, x_n)$ be finite.

For a cone with vertex $x_1^{a_1} \cdots x_n^{a_n} \in G$

x_i is a *multiplicative variable* iff

$a_i = \max\{\, b_i \mid x^b \in G;\ b_j = a_j \ \forall\, j < i \,\}$



Janet division

For $v = x_1^{a_1} \cdots x_n^{a_n}$:

$$x_i \in \mu \iff a_i = \max\{\, b_i \mid x^b \in G;\ b_j = a_j\ \forall\, j < i \,\}.$$

Example: $G = \{\, y\,z,\ x\,y\,z,\ x^2\,y\,z,\ x^2\,y^2 \,\}$

$y\,z$	$*$	y	z
$x\ \ y\,z$	$*$	y	z
$x^2\,y\,z$	x	$*$	z
$x^2\,y^2$	x	y	z

Thomas decomposition for nonlinear PDE systems

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

algebraic reduction:

$$p = u_{x,x,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$p \;\rightarrow\; r = c \cdot p - u_{x,x,y} \cdot q$$

differential reduction:

$$p = u_{x,x,y,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$\partial_y\, q = \frac{\partial q}{\partial u_{x,x,y}}\, u_{x,x,y,y} + \ldots$$

$$p \;\rightarrow\; r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

reduction requires: initial $c \neq 0$ and separant $\frac{\partial q}{\partial u_{x,x,y}} \neq 0$

Thomas decomposition for nonlinear PDE systems

$S = \{ p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \}$

Def. *Thomas decomposition* of diff. system S (or $\mathrm{Sol}(S)$):

$\mathrm{Sol}(S) = \mathrm{Sol}(S_1) \uplus \ldots \uplus \mathrm{Sol}(S_r)$, S_i simple diff. system

Def. S is *simple* if

(a) $p_1, \ldots, p_s, q_1, \ldots, q_t$ have pairwise distinct leaders,

(b) leading coefficients and discriminants of p_i and q_j do not vanish,

(c) $p_1, \ldots, p_s$ form a passive PDE system,

(d) $q_1, \ldots, q_t$ are reduced modulo $p_1, \ldots, p_s$.

set of *admissible derivations* $\mu_i \subseteq \{\partial_1, \ldots, \partial_n\}$ for p_i, $i = 1, \ldots, s$

Thomas decomposition for nonlinear PDE systems

$R = K\{u_1, \ldots, u_m\}$

Def. *Thomas decomposition* of diff. system S (or $\mathsf{Sol}(S)$):

$\mathsf{Sol}(S) = \mathsf{Sol}(S_1) \uplus \ldots \uplus \mathsf{Sol}(S_r)$, S_i simple diff. system

Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E diff. ideal generated by $p_1, \ldots, p_s$

q product of initials and separants of all p_i

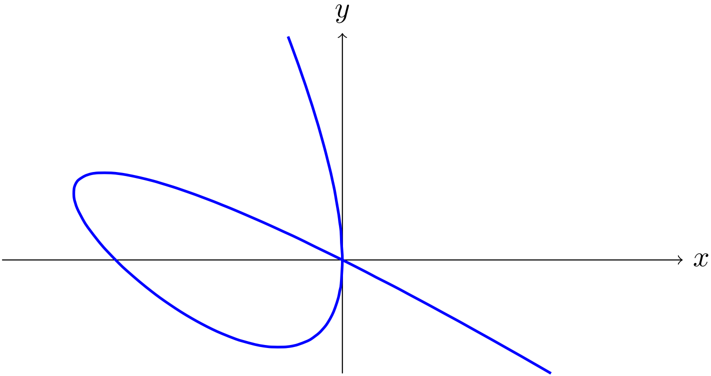
Then

$$E : q^\infty := \{ p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \} = \mathcal{I}_R(\mathsf{Sol}(S))$$

consists of all diff. polynomials in R vanishing on $\mathsf{Sol}(S)$.

Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas Decomposition

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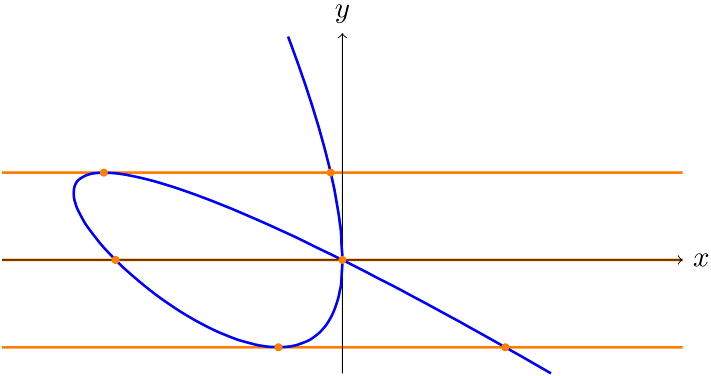

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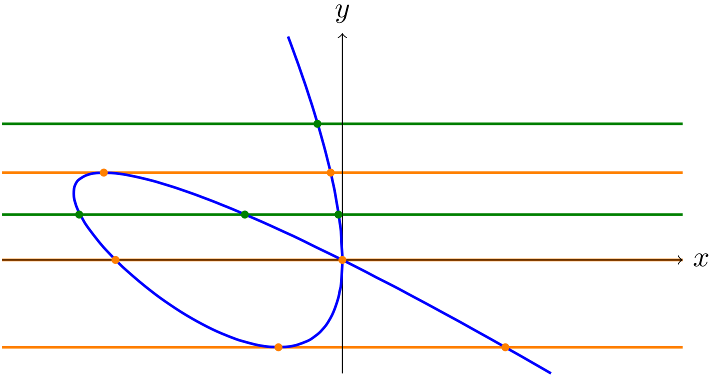

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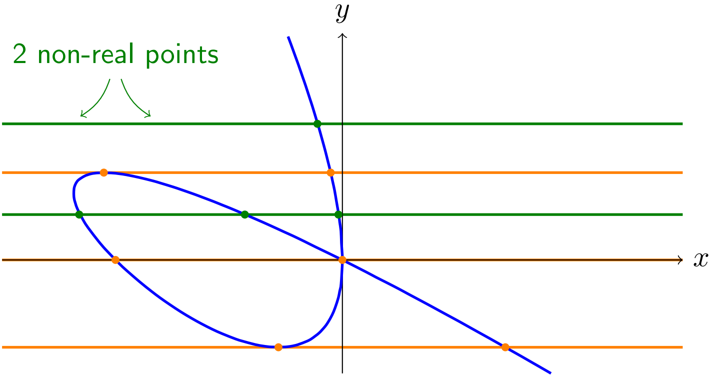


$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c = 0$	$2a\underline{x} + b = 0$	$b\underline{x} + c = 0$	
$4a\underline{c} - b^2 \neq 0$	$4a\underline{c} - b^2 = 0$		$c = 0$
		$b \neq 0$	$b = 0$
$a \neq 0$	$a \neq 0$	$a = 0$	$a = 0$
$x_1 \neq x_2$	$x_1 = x_2$	x_1	all $x \in \overline{\mathbb{Q}}$

solve $p(x) = 0$ for fixed choice of a, b, c

Thomas Decomposition

$$\left\{ \begin{array}{rcl} x^2 + y^2 - 1 & = & 0 \\ (1-y)\,t - x & = & 0 \end{array} \right. \qquad\qquad x > y > t$$

$$p_1 \;:=\; x^2 + y^2 - 1, \qquad p_2 \;:=\; x + ty - t \quad \in \quad \mathbb{Q}[x, y, t]$$

$$p_1 - (x - ty + t)\,p_2 \;=\; (y-1)\,((1+t^2)y - t^2 + 1)$$

Thomas decomposition:

$$\boxed{\begin{array}{rcl} (1+t^2)\underline{x} - 2t & = & 0 \\ (1+t^2)\underline{y} - t^2 + 1 & = & 0 \\ \underline{t}^2 + 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y} - 1 & = & 0 \end{array}}$$

Singular Solutions

$$p = \dot{y}^2 - 4t\dot{y} - 4y + 8t^2 = 0 \qquad p \in \mathbb{Q}(t)\{y\}$$

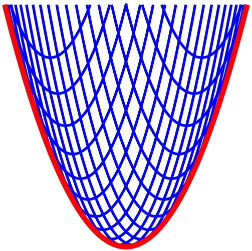
Separant of p: $\frac{\partial p}{\partial \dot{y}} = 2\dot{y} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{y}}, \dot{y}) = -16y + 16t^2$

Thomas decomposition:

$p = 0$
$y - t^2 \neq 0$

$y - t^2 = 0$



general solution: $y(t) = 2((t+c)^2 + c^2), \qquad c \in \mathbb{R}$

essential singular solution: $y(t) = t^2$

Thomas Decomposition

- 1937: J. M. Thomas: "Differential Systems".
- 1998: D. Wang: implementation for algebraic systems
- 2007: V. Gerdt: "On decomposition of algebraic PDE systems into simple subsystems"
- 2009: W. Plesken: "Counting solutions of polynomial systems via iterated fibrations"
- since 2009: implementations in Maple for
  - algebraic systems (T. Bächler)
  - systems of PDEs (M. Lange-Hegermann)

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. R.,
Algorithmic Thomas decomposition of algebraic and differential systems,
J. Symbolic Computation 47(10):1233–1266, 2012.

Thomas Decomposition

Example.

Thomas decomposition of $\{u_t - 6uu_x + \underline{u_{x,x,x}} = 0,\ u\underline{u_{t,x}} - u_t u_x = 0\}$:

$u = 0 \quad \{\partial_t, \partial_x\}$	$\underline{u_t} - 6uu_x = 0 \quad \{\partial_t, \partial_x\}$ $u_{x,x} = 0 \quad \{*, \partial_x\}$ $u \neq 0$	$u_t = 0 \quad \{\partial_t, \partial_x\}$ $\underline{u_{x,x,x}} - 6uu_x = 0 \quad \{*, \partial_x\}$ $u_{x,x} \neq 0$ $u \neq 0$

Generation of Difference Schemes

Laplace equation $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \rightsquigarrow \quad \oint_\Gamma -u_y dx + u_x dy = 0$

midpoint integration $\rightsquigarrow$

$$\begin{cases} (\theta_x \theta_y^2 - \theta_x) \circ u_y + (\theta_x^2 \theta_y - \theta_y) \circ u_x = 0\,, \\ 2\triangle h\, \theta_x \circ u_x - (\theta_x^2 - 1) \circ u = 0\,, \\ 2\triangle h\, \theta_y \circ u_y - (\theta_y^2 - 1) \circ u = 0\,. \end{cases}$$

$$\begin{cases} \theta_x \circ u_x - \frac{1}{2\triangle h}\,(\theta_x^2 - 1) \circ u = 0\,, \\ \theta_y \circ u_x + \theta_x \circ u_y - \frac{1}{2\triangle h}\,(\theta_x \theta_y ((\theta_x^2 - 1) + (\theta_y^2 - 1))) \circ u = 0\,, \\ \theta_x^2 \circ u_y - \frac{1}{2\triangle h}\,(\theta_x^2 \theta_y ((\theta_x^2 - 1) + (\theta_y^2 - 1)) - \theta_y (\theta_x^2 - 1)) \circ u = 0\,, \\ \theta_y \circ u_y - \frac{1}{2\triangle h}\,(\theta_y^2 - 1) \circ u = 0\,, \\ \frac{1}{2\triangle h}\,(\theta_x^4 \theta_y^2 + \theta_x^2 \theta_y^4 - 4\theta_x^2 \theta_y^2 + \theta_x^2 + \theta_y^2) \circ u = 0\,. \end{cases}$$

V. P. Gerdt, Y. A. Blinkov, V. V. Mozzhilkin,
Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations,
Symmetry, Integrability and Geometry: Methods and Applications, 2006.

2. Difference algebra

Difference algebra

Difference algebra (Ritt, Cohn, Levin, ...)

$\mathbb{Q} \subseteq K$ a difference field with commuting automorphisms δ_1, ..., δ_n

Difference polynomial ring with comm. endomorphisms δ_1, ..., δ_n

$K\{u\} := K[\delta_1^{i_1} \cdots \delta_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n], \qquad u_{(i_1,\ldots,i_n)} := \delta_1^{i_1} \cdots \delta_n^{i_n} u$

$K\{u\}$ not Noetherian (e.g., $[uu_1, uu_2, uu_3, \ldots] \subseteq K\{u\}$ not fin. gen.)

A difference ideal I of $K\{u\}$ is said to be

- *reflexive* if $\delta(p) \in I$ implies $p \in I$ $\qquad (p \in K\{u\})$
- *perfect* if $\delta_1^{i_1}(p)^{k_1} \ldots \delta_r^{i_r}(p)^{k_r} \in I$ implies $p \in I$

Difference algebra

Thm. (Ritt-Raudenbush).
Every perfect difference ideal of $K\{u_1, \ldots, u_m\}$ is finitely generated and is intersection of finitely many reflexive prime difference ideals.

Prop. The vanishing ideal in $K\{u_1, \ldots, u_m\}$ of a set of m-tuples of functions is a perfect difference ideal.

Thm. (Difference Nullstellensatz; Trushin, preprint, 2009).
Every difference field embeds into a difference closed field.

Thm. (Cohn)
Existence of generic zeros of reflexive prime difference ideals in piecewise analytic functions on $\mathbb{R}_+$

Difference algebra

Shall we assume reflexivity?

$$\left\{ \begin{array}{rcl} u(x+1) - u(x) & \neq & 0, \\ u(x+2) - u(x+1) & = & 0 \end{array} \right.$$

Under the assumption of reflexivity we have the consequence

$$u(x+1) - u(x) = 0, \qquad \text{contradiction}$$

Otherwise, e.g., if solutions $u \colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$ are to be found,

$$u(x) = \left\{ \begin{array}{ll} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0 \end{array} \right.$$

is a solution. However, there is no solution $u \colon \mathbb{Z} \to \mathbb{C}$.

Factorization

Examples.

a) $(u(x+3)-u(x))\,(u(x+2)-u(x))\,(u(x+1)-u(x))=0$

splitting:

$$\begin{cases} u(x+3)-u(x)=0, \\ u(x+2)-u(x)\neq 0, \\ u(x+1)-u(x)\neq 0 \end{cases}$$

$$\begin{cases} u(x+2)-u(x)=0, \\ u(x+1)-u(x)\neq 0 \end{cases}$$

$$\{\; u(x+1)-u(x)=0$$

b) (Ritt, Doob, Cohn)

$$P=(u(x+1)-u(x))^2-2(u(x+1)+u(x))+1=0$$

is algebraically irreducible; but solutions in meromorphic functions:

$(x+c(x))^2$, $(c(x)\exp(\pi i x)+1/2)^2$, where $c(x+1)=c(x)$

$\delta P-P=(u(x+2)-u(x))\,(u(x+2)-2u(x+1)+u(x)-2)$

Factorization

Examples.

c) (Ritt)

$$\begin{cases} u(x)^2 + 1 = 0, \\ u(x+1) - u(x) = 0, \\ v(x)^2 + 1 = 0, \\ v(x+1) + v(x) = 0. \end{cases}$$

A consequence of this system is

$$u(x)^2 - v(x)^2 = 0,$$

hence

$$(u(x) - v(x))\,(u(x) + v(x)) = 0.$$

Neither of the two systems obtained by the case splitting admits a solution.

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Trans. Amer. Math. Soc. 46:445–452, 1939.

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Nuclear Instruments and Methods in Physics Research A, 559 (1), 2006, pp. 215–219

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Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification,
in: S. M. Watt (ed.), Proceedings of ISSAC 2010, TU München, Germany, pp. 53–59

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Computer Science Journal of Moldova, 20 (2), 2012, pp. 203–226

3. Towards a symbolic package for nonlinear difference equations

Example

$$\begin{cases} u(x+2) - u(x) &= 0, \\ u(x+1)^2 - u(x) &= 0 \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} u(x)^4 - u(x) &= 0, & \{*\} \\ u(x+1) - u(x)^2 &= 0 & \{\delta\} \end{cases}$$

or

$$\begin{cases} u(x) &= 0, \\ & \end{cases} \qquad \begin{cases} u(x)^3 - u(x) &= 0, & \{*\} \\ u(x+1) - u(x)^2 &= 0 & \{\delta\} \end{cases}$$

Solutions:

$$\begin{cases} u(x) = \text{primitive 3rd root of unity,} \\ u(x+1) = \text{other primitive 3rd root of unity, alternating} \end{cases}$$

Input: $L \subset R - K$ finite and a ranking $>$ on R such that $L = S^{=}$ for some finite difference system S which is simple as an algebraic system (in the finitely many indeterminates $(u_k)_J$ which occur in it, totally ordered by $>$, not necessarily square-free)

Output: $a \in \{\textbf{true}, \textbf{false}\}$ and $L' \subset R - K$ finite such that

$$\langle\, L' \,\rangle : q^{\infty} = \langle\, L \,\rangle : q^{\infty}, \qquad q := \prod_{\substack{p \in L, \\ \theta \in \mathrm{ld}(L - \{p\}) : \mathrm{ld}(p)}} \mathrm{init}(\theta p), \tag{1}$$

and, in case $a = \textbf{true}$, there exist no p_1, $p_2 \in L'$, $p_1 \neq p_2$, such that we have $v := \mathrm{ld}(p_1) = \theta\,\mathrm{ld}(p_2)$ for some $\theta \in \mathrm{Mon}(\Delta)$ and $\deg_v(p_1) \geq \deg_v(\theta p_2)$

Algorithm:

$L' \leftarrow L$
while there exist $p_1, p_2 \in L'$, $p_1 \neq p_2$ and $\theta \in \mathrm{Mon}(\Delta)$ such that we have $v := \mathrm{ld}(p_1) = \theta\,\mathrm{ld}(p_2)$ and $\deg_v(p_1) \geq \deg_v(\theta p_2)$ **do**
  $L' \leftarrow L' - \{p_1\}$; $v \leftarrow \mathrm{ld}(p_1)$
  $r \leftarrow \mathrm{init}(\theta p_2) \cdot p_1 - \mathrm{init}(p_1) \cdot v^d \cdot \theta p_2$, where $d := \deg_v(p_1) - \deg_v(\theta p_2)$
  if $r \neq 0$ **then**
    return $(\textbf{false}, L' \cup \{r\})$
  end if
end while
return $(\textbf{true}, L')$

Input: $r \in R$, $T = \{ (p_1, \mu_1), (p_2, \mu_2), \ldots, (p_s, \mu_s) \}$, and a ranking $>$ on R, where T is Janet complete (with respect to $>$)

Output: $r' \in R$ and an element b of the multiplicatively closed set generated by

$$\bigcup_{i=1}^{s} \{ \theta \operatorname{init}(p_i) \mid \theta \in \operatorname{Mon}(\Delta),\ \theta \operatorname{ld}(p_i) < \operatorname{ld}(r) \} \cup \{1\}$$

such that r' is Janet reduced modulo T, and such that $r' = r$, $b = 1$ if $T = \emptyset$, and $r' + \langle p_1, \ldots, p_s \rangle = b \cdot r + \langle p_1, \ldots, p_s \rangle$ otherwise

Algorithm:

$r' \leftarrow r; \quad b \leftarrow 1$
if $r' \notin K$ **then**
    $v \leftarrow \operatorname{ld}(r')$
    while $r' \notin K$ and there exist $(p, \mu) \in T$ and $\theta \in \operatorname{Mon}(\mu)$ such that $v = \theta \operatorname{ld}(p)$ and $\deg_v(r') \geq \deg_v(\theta p)$ **do**
        $r' \leftarrow \operatorname{init}(\theta p) \cdot r' - \operatorname{init}(r') \cdot v^d \cdot \theta p$, where $d := \deg_v(r') - \deg_v(\theta p)$
        $b \leftarrow \operatorname{init}(\theta p) \cdot b$
    end while
    for each coefficient c of r' (as a polynomial in v) **do**
        $(r'', b') \leftarrow$ `Janet-reduce`(c, T, $>$)
        replace the coefficient $b' \cdot c$ in $b' \cdot r'$ with r'' and replace r' with this result
        $b \leftarrow b' \cdot b$
    end for
end if

Input: A finite difference system S over R, a ranking $>$ on R, and a total ordering on Δ (used by `Decompose`)

Output: A decomposition of S

Algorithm:

$Q \leftarrow \{S\}$; $T \leftarrow \emptyset$
repeat
  choose $L \in Q$ and remove L from Q
  $L \leftarrow L \cup \{\, q \neq 0 \mid q \in (\mathrm{Mon}(\Delta)\, L^{\neq}) \cap K[V] \,\}$, where $K[V]$ is the smallest K-subalgebra of R containing $L^{=}$
  compute a Thomas decomposition $\{\, A_1, \ldots, A_r \,\}$ of L as an algebraic system
  for $i = 1, \ldots, r$ **do**
    if $A_i = \emptyset$ **then** *// no equation and no inequation*
      return $\{\emptyset\}$
    else
      $(a, G) \leftarrow$ `Auto-reduce`$(A_i^{=}, >)$
      if $a =$ **true then**
        ...
      else
        insert $\{\, p = 0 \mid p \in G \,\} \cup \{\, q \neq 0 \mid q \in A_i^{\neq} \,\}$ into Q
      end if
    end if
  end for
until $Q = \emptyset$
return T

Input: A finite difference system S over R, a ranking $>$ on R, and a total ordering on Δ (used by `Decompose`)

Output: A decomposition of S

Algorithm:

...

$(a, G) \leftarrow$ `Auto-reduce`$(A_i^{=}, >)$

if $a =$ **true then**

    $J \leftarrow$ `Decompose`(G)

    $P \leftarrow \{\, \mathrm{NF}(\delta\, p, J, >) \mid (p, \mu) \in J,\ \delta \in \overline{\mu} \,\}$

    if $P \subseteq \{0\}$ **then** *// J is passive*

        replace each inequation $q \neq 0$ in A_i with $\mathrm{NF}(q, J, >) \neq 0$

        if $0 \notin A_i^{\neq}$ **then**

            insert $\{\, p = 0 \mid (p, \mu) \in J \,\} \cup \{\, q \neq 0 \mid q \in A_i^{\neq} \,\}$ into T

        end if

    else if $P \cap K \subseteq \{0\}$ **then**

        insert $\{\, p = 0 \mid (p, \mu) \in J \,\} \cup \{\, p = 0 \mid p \in P - \{0\} \,\} \cup \{\, q \neq 0 \mid q \in A_i^{\neq} \,\}$ into Q

    end if

else

    insert $\{\, p = 0 \mid p \in G \,\} \cup \{\, q \neq 0 \mid q \in A_i^{\neq} \,\}$ into Q

end if

...

Example

$$\left\{ \begin{array}{rcl} (u(x+1,y)-u(x,y+2))\,u(x+1,y+1)+u(x+1,y+1)^2 &=& 0, \\ u(x+1,y+1)-u(x,y) &=& 0 \end{array} \right.$$

$$u(x,y)=0 \quad \left\{ \begin{array}{rcl} u(x,y) &\neq& 0, \\ \underline{u(x+1,y+1)}-u(x,y) &=& 0, \\ -u(x,y)\,\underline{u(x,y+2)}+u(x,y)\,u(x+1,y)+u(x,y)^2 &=& 0 \end{array} \right.$$

consider

$$-u(x+1,y)\,\underline{u(x+1,y+2)}+u(x+1,y)\,u(x+2,y)+u(x+1,y)^2=0$$

second simple system:

$$\left\{ \begin{array}{rcl} \underline{u(x+1,y+1)}-u(x,y) &=& 0, \\ -u(x,y)\,\underline{u(x,y+2)}+u(x,y)\,u(x+1,y)+u(x,y)^2 &=& 0, \\ -u(x+1,y)\,u(x,y+1)+u(x+1,y)\,\underline{u(x+2,y)}+u(x+1,y)^2 &=& 0, \\ u(x,y) &\neq& 0, \\ u(x+1,y) &\neq& 0 \end{array} \right.$$

Example

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$$
\left\{
\begin{array}{l}
\frac{u(n,j+1,k)-u(n,j-1,k)}{2h} + \frac{v(n,j,k+1)-v(n,j,k-1)}{2h} = 0, \\
\frac{u(n+1,j,k)-u(n,j,k)}{\tau} + \frac{u(n,j+1,k)^2-u(n,j-1,k)^2}{2h} + \\
\qquad \frac{u(n,j,k+1)v(n,j,k+1)-u(n,j,k-1)v(n,j,k-1)}{2h} + \frac{p(n,j+1,k)-p(n,j-1,k)}{2h} - \\
\qquad \frac{1}{re}\ \frac{(u(n,j+1,k)-2u(n,j,k)+u(n,j-1,k))+(u(n,j,k+1)-2u(n,j,k)+u(n,j,k-1))}{h^2} = 0, \\
\frac{v(n+1,j,k)-v(n,j,k)}{\tau} + \frac{u(n,j+1,k)v(n,j+1,k)-u(n,j-1,k)v(n,j-1,k)}{2h} + \\
\qquad \frac{v(n,j,k+1)^2-v(n,j,k-1)^2}{2h} + \frac{p(n,j,k+1)-p(n,j,k-1)}{2h} - \\
\qquad \frac{1}{re}\ \frac{(v(n,j+1,k)-2v(n,j,k)+v(n,j-1,k))+(v(n,j,k+1)-2v(n,j,k)+v(n,j,k-1))}{h^2} = 0, \\
\frac{u(n,j+2,k)^2-2u(n,j,k)^2+u(n,j-2,k)^2}{4h^2} + \frac{v(n,j,k+2)^2-2v(n,j,k)^2+v(n,j,k-2)^2}{4h^2} + \\
\qquad 2\ \frac{u(n,j+1,k+1)v(n,j+1,k+1)-u(n,j+1,k-1)v(n,j+1,k-1)}{4h^2} - \\
\qquad 2\ \frac{u(n,j-1,k+1)v(n,j-1,k+1)+u(n,j-1,k-1)v(n,j-1,k-1)}{4h^2} + \\
\qquad \frac{p(n,j+2,k)-2p(n,j,k)+p(n,j-2,k)}{4h^2} + \frac{p(n,j,k+2)-2p(n,j,k)+p(n,j,k-2)}{4h^2} = 0
\end{array}
\right.
$$

Reflexivity

$$(*) \qquad \begin{cases} \underline{u(x+2)} - u(x) &= 0, \\ \underline{v(x+1)} - u(x) &= 0 \end{cases}$$

Consequence: $u(x+2) - v(x+1) = 0$

Under the assumption of reflexivity: $u(x+1) - v(x) = 0$, hence,

$$\begin{cases} \underline{u(x+1)} - v(x) &= 0, \\ \underline{v(x+1)} - u(x) &= 0 \end{cases}$$

Algebraic Thomas decomposition of $(*)$ with ranking $u(x+2) < v(x+1) < u(x)$:

$$\begin{cases} \underline{v(x+1)} - u(x+2) &= 0, \\ \underline{u(x)} - u(x+2) &= 0 \end{cases}$$

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