

On finite difference approximations to Kortevég-de Vries equation and to its conservation laws

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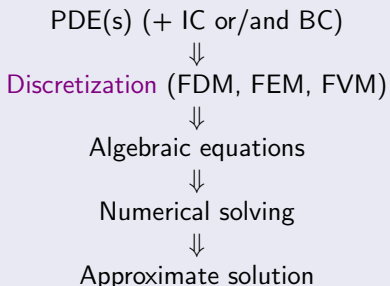
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Plan

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Research problem

Solving PDE(s) in practice:



The main research problem here is to construct a **compatible (mimetic) discretization** which **inherits within the required accuracy** such fundamental properties of the PDE(s) as **conservation laws**, symmetries, maximum principle, etc.

Approach to discretization (Gerdt, Blinkov, Mozzhilkin'2006)

- 1 Choose a regular grid.
- 2 For an input PDE system **complete it to involution**.
- 3 If possible, convert PDE into **integral conservation law form** with a polynomial integrand and choose a control volume (**integration contour/surface**).
- 4 **Add** to the output of the previous step the exact **integral relations** between derivatives of dependent variables that occur in the output of Step 2.
- 5 To discretize the obtained equations **apply numerical integration**.
- 6 **Eliminate partial derivatives of dependent variables** from the obtained difference equations by applying the Gröbner bases technique. This yields FDA to PDE(s) (**difference scheme**) as the subset of the Gröbner basis that does not contain partial derivatives.

Evolution equations

Let ∂_x be the derivation operator and $\mathcal{R} := \mathbb{Q}(\alpha, \beta, \dots)\{u\}$ be the **ordinary differential polynomial ring** over the field $\mathbb{Q}(\alpha, \beta, \dots)$ of constants (parameters).

Here we consider quasilinear evolution equations (QLEE) of the form

$$u_t = a u_m + F(u_{m-1}, \dots, u_1, u), \quad 0 \neq a \in \mathbb{Q}, \quad m \in \mathbb{N}_{>0}$$

where $u_k := \partial_x^k u$ ($0 \leq k \leq m$), $u_0 := u$ and $F \in \mathcal{R}$ is a differential polynomial of order $m - 1$ in ∂_x (denotation: $\text{ord}(F) = m - 1$).

If there is a differential polynomial $P \in \mathcal{R}$ such that $F = \partial_x P$ then the equation admits the **differential conservation law form**

$$u_t = \partial_x (a u_{m-1} + P), \quad P \in \mathcal{R}, \quad \text{ord}(P) = m - 2$$

Examples

The set of QLEEs contains most of classical equations (Kudryashov'10):

- Kortevveg-de Vries (KdV) hierarchy

$$u_t + u_{xxx} + 6uu_x = 0 \quad (\text{KdV})$$

$$u_t + u_{xxxxx} + 10u_{xx} + 30u^2u_x + 20u_xu_{xx} = 0$$

.....

- Burgers hierarchy (Burgers equation for $n = 1$)

$$u_t + a \partial_x (\partial_x + u)^n u = 0, \quad a \neq 0, \quad n \in \mathbb{N}_{\geq 0}$$

- Kuramoto-Sivashinsky equation

$$u_t + uu_x + a u_{xx} + b u_{xxx} + c u_{xxxx} = 0$$

- Burgers-Huxley equation

$$u_t + a u_{xx} + b uu_x + c u + \delta u^2 + d u^3 = 0$$

and their various generalizations.

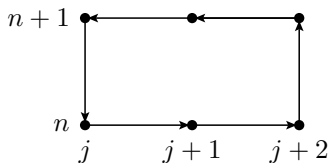
Discretization of QLEE I

If a QLEE admits the differential polynomial conservation law form we convert it into the **equivalent integral conservation law form**

$$\oint_{\Gamma} (P + au_{m-1}) dt + u dx = 0$$

where Γ is an arbitrary singly connected integration contour.

We choose the **Cartesian grid** with $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and choose simple rectangular integration contour as a **“control volume”**



Discretization of QLEE II

To discretize the contour integral we apply a numerical method of its evaluation in term of the **grid function**

$$u_j^n := u(t_n, x_j)$$

Then we add the (exact) integral relations

$$\int_{x_j}^{x_{j+1}} u_{k+1} dx = u_k(t, x_{j+1}) - u_k(t, x_j), \quad k = 1, \dots, m-2$$

and approximate the integrals numerically on the grid.

As a result, **we obtain a system of difference equations** containing

$$u_j^n, u_{1j}^n, \dots, u_{m-1j}^n$$

Finally, **difference elimination of partial derivatives yields FDA.**

Discretization of KdV I

To be specific, we consider the Korteweg-de Vries equation (KdV) in the following form (with $P = \alpha u^2/2$)

$$f = 0, \quad f := u_t + \alpha u u_x + \beta u_{xxx}, \quad u = u(t, x), \quad \alpha, \beta \in \mathbb{R}.$$

Its **integral conservation law form** reads

$$u_t + (P + \beta u_{xx})_x = 0 \iff \oint_{\Gamma} -(P + \beta u_{xx}) dt + u dx = 0$$

The additional integral relations are given by

$$\int_{x_j}^{x_{j+2}} u_{xx} dx = u_x(t, x_{j+2}) - u_x(t, x_j),$$

$$\int_{x_j}^{x_{j+1}} u_x dx = u(t, x_{j+1}) - u(t, x_j).$$

Discretization of KdV II

To approximate numerically the contour integral, we apply the **trapezoidal rule** to integration over t and the **midpoint rule** to integration over x .

For numerical approximations of the integral relations we apply the **trapezoidal rule** for the integration of u_x and the **midpoint rule** for the integration of u_{xx} . This leads to the difference approximation to KdV which is outputted by the following MAPLE code ([Gerdt, Blinkov, Marinov'17](#))

Maple code

```

> restart:
> with(LDA):
> L:= [ (- (P(n,j)+P(n+1,j)-P(n,j+2)-P(n+1,j+2)) - (beta*uxx(n,j)+beta*uxx(n+1,j)
  -beta*uxx(n,j+2)-beta*uxx(n+1,j+2))) *tau/2+(u(n+1,j+1)-u(n,j+1))*2*h,
  (ux(n,j+1)+ux(n,j))*h/2-(u(n,j+1)-u(n,j)),
  2*uxx(n,j+1)*h-(ux(n,j+2)-ux(n,j)) ]:
> JanetBasis(L, [n,j], [uxx,ux,u,F], 2):
> collect(%[1,1]/(4*tau*h**3), [tau,h]);

```

$$\frac{\frac{1}{4} P(n+1, j+3) + \frac{1}{4} P(n, j+3) - \frac{1}{4} P(n+1, j+1) - \frac{1}{4} P(n, j+1)}{h} + \frac{1}{h^3} \left(\frac{1}{4} \beta u(n+1, j) \right.$$

$$\left. + 4 - \frac{1}{2} \beta u(n+1, j+3) + \frac{1}{4} \beta u(n, j+4) - \frac{1}{2} \beta u(n, j+3) + \frac{1}{2} \beta u(n+1, j+1) \right.$$

$$\left. - \frac{1}{4} \beta u(n+1, j) + \frac{1}{2} \beta u(n, j+1) - \frac{1}{4} \beta u(n, j) \right) + \frac{u(n+1, j+2) - u(n, j+2)}{\tau}$$

Computer algebra software used

To perform algebraically the difference elimination of the grid functions, that correspond to partial derivatives of u , from the obtained discrete system we use the Maple package **LDA (Linear Difference Algebra)**.

LDA created by **D.Robertz** (RWTH, Aachen) is freely available (<http://wwwb.math.rwth-aachen.de/Janet/>). It implements the involutive algorithm (**Gerdt,Blinkov'98**) based on Janet division and **specialized to difference ideals generated by linear difference polynomials**.

Note that to apply LDA we “hide” the nonlinearity (caused by the presence of u^2 in the input difference equations) into the extra grid function $P_j^n := \alpha(u_j^n)^2/2$.

Numerical solving of KdV

In the conventional notations the obtained discretization (**difference scheme**) reads

$$\tilde{f} = 0, \quad \text{where} \quad \tilde{f} := \frac{u_j^{n+1} - u_j^n}{\tau} + \frac{(P_{j+1}^{n+1} - P_{j-1}^{n+1}) + (P_{j+1}^n - P_{j-1}^n)}{4h} \\ + \frac{\beta(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + \beta(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n)}{4h^3}.$$

To construct a numerical solution to KdV we apply the **method of simple iteration** and exploit the following approximation

$$v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx v_{k+1} \cdot 2v_k - v_k^2.$$

Properties of the FDA (scheme) derived I

The difference polynomial \tilde{f} generates the **perfect difference ideal** $[[\tilde{f}]]$ in the **inversive difference polynomial ring** with differences $\sigma_t, \sigma_x, \sigma_t^{-1}, \sigma_x^{-1}$ where

$$\sigma_t \circ \tilde{f}_j^n = \tilde{f}_j^{n+1}, \quad \sigma_x \circ \tilde{f}_j^n = \tilde{f}_{j+1}^n, \quad \sigma_t^{-1} \circ \tilde{f}_j^n = \tilde{f}_j^{n-1}, \quad \sigma_x^{-1} \circ \tilde{f}_j^n = \tilde{f}_{j-1}^n.$$

The consistency of $\tilde{f} = 0$ with $f = 0$ implies that **every element in** $[[f]]$ **is approximated by an element in** $[[\tilde{f}]]$ (**strong consistency**).

Theorem (Gerdt'12)

A FDA \tilde{F} to a (system of) differential polynomial(s) F is s -consistent iff every element in the Gröbner (standard) basis of the difference ideal generated by \tilde{F} provides FDA to an element of the radical differential ideal generated by F .

Properties of the FDA (scheme) derived II

Using the library SYMPY (<http://www.sympy.org/en/>) written in PYTHON we computed the modified equation for our scheme

$$\begin{aligned}
 & u_t + \alpha uu_1 + \beta u_3 + h^2 \left[\alpha \left(\frac{1}{6} uu_3 + \frac{1}{2} u_1 u_2 \right) + \frac{\beta}{4} u_5 \right] + \\
 & + \tau^2 \left[\alpha^3 \left(\frac{1}{12} u_3 + \frac{3}{4} u^2 u_1 u_2 + \frac{1}{2} uu_1^3 \right) + \alpha^2 \beta \left(\frac{1}{4} u^2 u_5 + \frac{7}{4} uu_1 u_4 + \right. \right. \\
 & \quad \left. \left. + \frac{11}{4} uu_2 u_3 + \frac{9}{4} u_1^2 u_3 + \frac{11}{4} u_1 u_2^2 \right) + \alpha \beta^2 \left(\frac{1}{4} uu_7 + u_1 u_6 + \right. \right. \\
 & \quad \left. \left. + \frac{9}{4} u_2 u_5 + \frac{7}{2} u_3 u_4 \right) + \frac{\beta^3}{12} u_9 \right] + \mathcal{O}(\tau^4, \tau^2 h^2, h^4)
 \end{aligned}$$

where

$$u_k := \underbrace{u_{xx \dots x}}_{k \text{ times}}, \quad k \geq 1.$$

Properties of the FDA (scheme) derived III

- 1 The modified equation shows that **the scheme has the 2-nd order** in τ and in h .
- 2 The scheme is implicit, and hence **it is unconditionally stable**.
- 3 Because of universally adopted condition for convergency of difference schemes (rigorously proved for linear Cauchy problem - the Lax(-Richtmyer) equivalence theorem):

$$\text{convergence} = \text{consistency} + \text{stability}$$

the obtained scheme is convergent.

Other schemes with $\mathcal{O}(\tau^2, h^2)$ approximation

Scheme I. The explicit scheme (Belashov, Vladimirov'05, Eq.1.80)

$$u_i^{n+1} = u_i^{n-1} - \frac{\alpha\tau}{h} u_i^n (u_{i+1}^n - u_{i-1}^n) - \frac{\beta\tau}{h^3} (u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n)$$

stable for $\tau \leq \frac{2h^3}{3\sqrt{3}\beta} \cong 0.384 \frac{h^3}{\beta}$. The modified equation

$$\begin{aligned} u_t + \alpha uu_1 + \beta u_3 + h^2 \left(\frac{\alpha}{6} uu_3 + \frac{\beta}{4} u_5 \right) - \\ - \tau^2 \left[\alpha^3 \left(\frac{1}{6} u^3 u_3 + \frac{3}{2} u^2 u_1 u_2 + uu_1^3 \right) + \alpha^2 \beta \left(\frac{1}{2} u^2 u_5 + \frac{7}{2} uu_1 u_4 + \right. \right. \\ \left. \left. + \frac{11}{2} uu_2 u_3 + \frac{9}{2} u_1^2 u_3 + \frac{11}{2} u_1 u_2^2 \right) + \alpha \beta^2 \left(\frac{1}{2} uu_7 + 2u_1 u_6 + \right. \right. \\ \left. \left. + \frac{9}{2} u_2 u_5 + 7u_3 u_4 \right) + \frac{\beta^3}{6} u_9 \right] + \mathcal{O}(\tau^4, h^4) \end{aligned}$$

Scheme II. The implicit scheme (Belashov, Vladimirov'05, Eq.1.96)

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{\alpha}{4h} \left[u_j^n \left(u_{j+1}^{n+1} - u_{j-1}^{n+1} \right) + u_j^{n+1} \left(u_{j+1}^n - u_{j-1}^n \right) \right] +$$

$$+ \frac{\beta}{4h^3} \left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1} + \right.$$

$$\left. + u_{j+2}^n - 2u_j^{n+1} + 2u_{j-1}^n - u_{j-2}^n \right) = 0.$$

The modified equation

$$u_t + \alpha uu_1 + \beta u_3 + h^2 \left(\frac{\alpha}{6} uu_3 + \frac{\beta}{4} u_5 \right) +$$

$$+ \tau^2 \left[\alpha^3 \left(\frac{1}{12} u^3 u_3 + \frac{1}{4} u^2 u_1 u_2 \right) + \alpha^2 \beta \left(\frac{1}{4} u^2 u_5 + \frac{5}{4} uu_1 u_4 + \right. \right.$$

$$\left. + \frac{9}{4} uu_2 u_3 + \frac{7}{4} u_1^2 u_3 + \frac{11}{4} u_1 u_2^2 \right) + \alpha \beta^2 \left(\frac{1}{4} uu_7 + u_1 u_6 + \right.$$

$$\left. + \frac{9}{4} u_2 u_5 + 3u_3 u_4 \right) + \frac{\beta^3}{12} u_9 \left. \right] + \mathcal{O}(\tau^4, \tau^2 h^2, h^4)$$

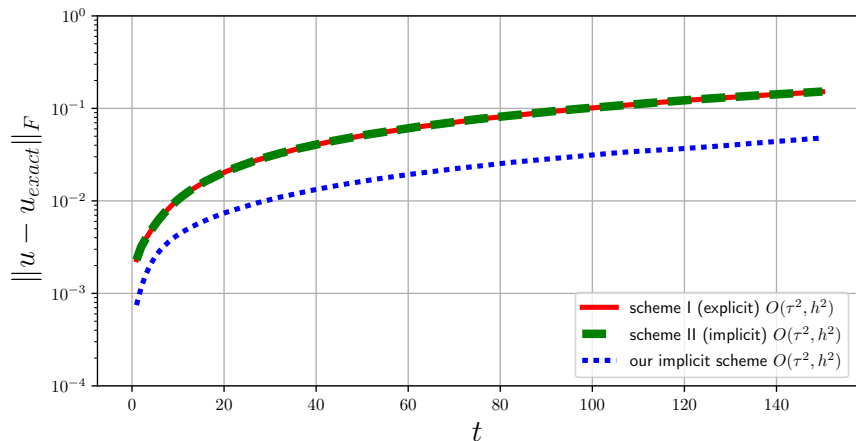
Computational experiment

Our numerical analysis of the above difference schemes was done with the Python package SCIPY (<http://scipy.org>). As a benchmark, we used the exact one-soliton solution

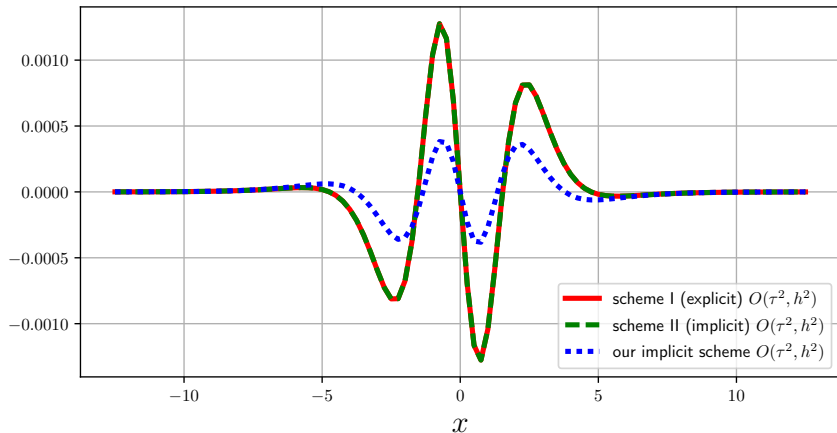
$$u_{\text{exact}}(t, x) = \frac{2k_1^2}{\cosh(k_1(x - 4k_1^2 t))^2}$$

to the KdV with $\alpha = 6$, $\beta = 1$ and $k_1 = 0.4$. In so doing, we fixed $h = 0.256$ $\tau = 0.1h$ and considered the solution in interval $-50 \leq x \leq 50$ with periodic boundary conditions (cf. Belashov, Vladimirov'05, p.49). The numerical inaccuracy was estimated by the Frobenius norm.

Numerical discrepancy



The left-hand side of modified equation for $u_{\text{exact}}(0, x)$



Conservation laws

The KdV equation

$$f = 0, \quad f := u_t + \alpha uu_x + \beta u_{xxx}, \quad u = u(t, x), \quad \alpha, \beta \in \mathbb{R}.$$

has infinitely many local conservation laws

$$\left\{ \underbrace{\partial_t T_i + \partial_x X_i}_{\mathcal{E}_i} = 0 \Big|_{f=0} \implies \frac{d}{dt} \int_{-\infty}^{\infty} T_i dx + [X_i]_{-\infty}^{\infty} = 0 \mid i \in \mathbb{N}_{\geq 1} \right\}$$

where $T_i = T_i(u)$ are densities and $X_i = X_i(u)$ are fluxes.

Conserved densities and fluxes

For $\alpha = 3$, $\beta = 1$ the low order conservation laws are given by

i	T_i and X_i
1	$T_1 = u \quad X_1 = 3u^2 + u_{xx}$
2	$T_2 = u_x \quad X_2 = 6uu_x + u_{xxx}$
3	$T_3 = u^2 + u_{xx} \quad X_3 = 5u_x^2 + 8uu_{xx} + 4u^3 + u_{xxxx}$
4	$T_4 = 4uu_x + u_{xxx}$ $X_4 = 18u_x u_{xx} + 24u^2 u_x + 10uu_{xxx} + u_{xxxxx}$
5	$T_5 = 2u^3 + 6uu_{xx} + 10u_x^2 + u_{xxxx}$ $X_5 = 42u^2 u_{xx} + 19u_{xx}^2 + 60uu_x^2 + 28u_x u_{xxx} + 12uu_{xxxx} + 9u^4 + u_{xxxxx}$
...

These conservation laws were computed by using Maple package PDEBELLII (Miao,Wang,Chen,Yang'14).

Conservation laws expressed via f

Conservation laws \mathcal{C}_i belong to the radical differential ideal $[f] \subset \mathcal{R} = \mathbb{Q}(\alpha, \beta)\{u\}$ generated by f . In particular,

i	\mathcal{C}_i	$\text{ord}_x(\mathcal{C}_i)$
1	f	3
2	f_x	4
3	$f_{xx} + 2uf$	5
4	$f_{xxx} + uf_x^4 + u_x^4 f$	6
5	$f_{xxxx} + 6uf_{xx} + 5u_x f_x + 6u_{xx} f + 6u^2 f$	7
...

\mathcal{C}_i were computed by using Maple package DIFFERENTIALTHOMAS (Bächler, Gerdt, Lange-Hegermann, Robertz'12).

FDA to conservation laws I

Theorem

For each conservation law \mathcal{E}_i of KdV there is an element \tilde{f}_i in the perfect difference ideal $[[\tilde{f}]]$ such that \tilde{f}_i approximates \mathcal{E}_i with the accuracy $O(\tau^2, h^2)$ corresponding to the accuracy of \tilde{f} .

We illustrate this theorem by the 3rd and 4th KdV conservation laws

$$\mathcal{E}_3 = f_{xx} + 2uf,$$

$$\mathcal{E}_4 = f_{xxx} + uf_x^4 + u_x^4 f.$$

FDA to conservation laws II

With regard to forward and backward differences

$$\Delta_p := \frac{1}{h} (\sigma_x - 1), \quad \Delta_m := \frac{1}{h} (1 - \sigma_x^{-1}).$$

we obtain

$$\frac{1}{2} (\Delta_p + \Delta_m) \circ u \xrightarrow{h \rightarrow 0} u_x + \mathcal{O}(h^2),$$

$$\frac{1}{2} (\Delta_p + \Delta_m) \circ f \xrightarrow{h \rightarrow 0} f_x + \mathcal{O}(h^2),$$

$$\Delta_m \Delta_p \circ f \xrightarrow{h \rightarrow 0} f_{xx} + \mathcal{O}(h^2),$$

$$\Delta_p \Delta_m \Delta_p \circ f - \frac{h}{2} \Delta_m \Delta_p \Delta_m \Delta_p \circ f \xrightarrow{h \rightarrow 0} f_{xxx} + \mathcal{O}(h^2).$$

Conclusions

- We considered a class of nonlinear evolution PDEs containing the classical KdV equation which has exact multi-soliton solutions and infinitely many conservation laws.
- We generated a FDA (implicit difference scheme) to KdV with $\mathcal{O}(\tau^2, h^2)$ approximation. The scheme is consistent and stable.
- The conservation laws of KdV are approximated with accuracy $\mathcal{O}(\tau^2, h^2)$ by difference consequences of the scheme.
- Experimental comparison, on the exact one-soliton solution, of the constructed scheme with some other schemes of the same accuracy reveals its numerical superiority.

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