Matrices over Differential-difference Algebras

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Let R be a ring and σ be a ring endomorphism of R. A σ-derivation on R is a map $\delta: R \longrightarrow R$ satisfying: $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in R$. The skew polynomial ring (also called Ore polynomial ring) $R[x;\sigma,\delta]$ over R is the set of usual polynomials in x over R, *i.e.*, $\{\sum r_i x^i \mid r_i \in R\}$, with usual "$+$" and

$$xr = \sigma(r)x + \delta(r), \quad \forall r \in R.$$

We refer to Cohn [1], Goodearl and Warfield [2], Levin[4], and van der Put and Singer[3] for more details and the related topics.

Matrices over skew polynomial rings (also called Ore matrices) have been studied for decades with many applications in other areas like control theory and engineering. In this talk, we focus on various generalized inverses of Ore matrices.

Let K be a ring with an involution "$*$". For $A \in K^{m\times n}$ and $X \in K^{n\times m}$, consider the following equations:

$$(i)\ AXA = A,\ \ (ii)\ XAX = X,\ \ (iii)\ (AX)^* = AX,\ \ (iv)\ (XA)^* = XA,$$

where A^* is the transpose conjugate of A. If a matrix $X \in K^{n\times m}$ satisfies (i), then X is called a $\{1\}$-inverse of A. A matrix $X \in K^{n\times m}$ satisfying both of (i) and (ii) is called a $\{1,2\}$-inverse of A, and so on. In particular, X satisfying $\{i, ii, iii, iv\}$ is called the Moore-Penrose inverse of A, denoted by $A^\dagger$. More generalized inverses of matrices like group inverses and Drazin inverses of matrices can be found in [5].

We first use Jacobson forms of Ore matrices to discuss $\{1\}$-inverses. One of theorems is as follows:

Theorem. For any $A \in R[x;\sigma,\delta]^{m\times n}$, A has a $\{1\}$-inverse over $R[x;\sigma,\delta]$ if and only if its Jacobson form equals $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, that is, there exist invertible matrices $P \in R[x;\sigma,\delta]^{m\times m}$ and $Q \in R[x;\sigma,\delta]^{n\times n}$ such that $A = P \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q$. Furthermore, if X is a $\{1\}$-inverse of A over $R[x;\sigma,\delta]$, then X can be written as $Q^{-1} \begin{bmatrix} I_r & W_2 \\ W_3 & W_4 \end{bmatrix} P^{-1}$, where W_2, W_3, W_4 are arbitrary matrices over $R[x;\sigma,\delta]$.

As applications of $\{1\}$-inverses, we discuss Roth theorems and generalized Sylvester matrix equation, for example,

Theorem. If Ore matrices A, B, C and D all have $\{1\}$-inverses over $R[x;\sigma,\delta]$, then the following statements are equivalent:

1. The matrix equation $AXB+CYD=E$ has solutions over $R[x;\sigma,\delta]$.
2. The matrix equations $AX_1+Y_1D=E$ and $X_2B+CY_2=E$ have solutions over $R[x;\sigma,\delta]$.
3. $\operatorname{rank}\begin{pmatrix} C & E & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & E \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D \end{pmatrix} = \operatorname{rank}\begin{pmatrix} C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D \end{pmatrix}$ over $R[x;\sigma,\delta]$.
4. The matrix equation $\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix} X_3 + Y_3 \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix} = \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ has solutions over $R[x;\sigma,\delta]$.

For Moore-Penrose inverses, assume that R is a division ring with an involution "$*$". We give the sufficient and necessary conditions for extending "$*$" to be an involution on $R[x;\sigma,\delta]$, and then prove the following theorems:

Theorem. For any $A\in R[x;\sigma,\delta]^{m\times n}$, $A^\dagger$ exists over $R[x;\sigma,\delta]$ if and only if A^*AA^* has a $\{1\}$-inverse over $R[x;\sigma,\delta]$, and $\operatorname{rank}(A)=\operatorname{rank}(AA^*)=\operatorname{rank}(A^*A)$. Moreover $X=A^*(A^*AA^*)^{(1)}A^*$ is the unique MP-inverse of A over $R[x;\sigma,\delta]$.

Theorem. For any $A\in R[x;\sigma,\delta]^{m\times n}$, if the Jacobson form of A is $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, i.e., there exist invertible matrices $P\in R[x;\sigma,\delta]^{m\times m}$ and $Q\in R[x;\sigma,\delta]^{n\times n}$ such that $A=P\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}Q$, where $P=\begin{bmatrix} P_1 & P_2 \end{bmatrix}$, $Q=\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$, $P_1\in R[x;\sigma,\delta]^{m\times r}$ is the first r columns of P and $Q_1\in R[x;\sigma,\delta]^{r\times n}$ is the first r rows of Q, then $A^\dagger$ exists over $R[x;\sigma,\delta]$ if and only if the Jacobson form of $P_1^*P_1Q_1Q_1^*$ is I_r.

As applications, we give the general solutions for the linear systems of differential-difference polynomials, and some types of matrix equations.

This is a joined work with Qiwei Feng.

References

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