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Bounds for Proto-Galois Groups

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In studying linear differential equations of the type $Y' = AY, A \in M_n(C(t))$, it is often important to investigate the algebraic or differential relations among the solutions. The benefit of obtaining such data is that it can be used to anticipate the computational power needed to express solutions. In [4], Kolchin made this precise by establishing a link between how solutions to $Y' = AY$ might be expressed and different properties of the corresponding differential Galois group, an object he constructed exactly to capture relations among the solutions. These differential Galois groups can be realized as linear algebraic groups. In fact, many algorithms to compute them have been developed since Kolchin's foundational discussions of these results in [4] and [5].

Kovacic [6] proposed an algorithm for second-order differential equations. Compoint and Singer also provided an algorithm in [1] that can be applied to equations of any order, if it is known in advance that the differential Galois group is reductive. A general algorithm for computing the differential Galois group was designed by Hrushovski [3]. Making this algorithm practical and understanding its complexity is an important challenge. Hrushovski conjectured that none of its steps would "require more than doubly exponential time." In [2], Feng expounded on the details of Hrushovski's original algorithm with differential-algebraic terminology and improved the algorithm. He also formally defined an object computed in the first step of the algorithm, a proto-Galois group. Such a group is an algebraic group, contains the differential Galois group, and the computation of it allows one to reduce the computation of the differential Galois group to the hyperexponential case, which is addressed by the algorithm in [1]. In Hrushovski's algorithm, a proto-Galois group is computed by making an ansatz based on an a priori bound for the degrees of defining polynomials of the group. Thus, such a bound is an essential part of the algorithm. Moreover, it also needed for understanding the complexity of the algorithm.

In [2], Feng showed that there exists a proto-Galois group defined by polynomials of degree at most sextuply exponential in n. Sun [7] utilized triangular sets in place of the Groebner bases used by Feng. This different choice of representation for a group leads to a bound triply exponential in n.

We adopt a different emphasis from both Feng and Sun. Instead of focusing on equations for the group's corresponding radical ideal, we take a more geometric approach and focus on equations that define a proto-Galois group as an algebraic variety in $GL_n(C)$. In conjunction with exploiting the structural theory of algebraic groups, this approach allows us to further improve on Feng's bound and thereby improve the algorithm. We obtain an explicit bound of the form $n^{O(n^4)}$.

We also assess the practicality of using Hrushovski's algorithm for $n = 2, 3$, the cases that arise most often in applications. We expect to determine tighter bounds than our general bound suggests for these cases. In fact, we have established a tighter bound for $n = 2$, for which our final bound is 6. We will discuss how we obtained this result. We will also discuss work in progress for $n = 3$ and extending our methods for $n = 2$ to those cases.

Keywords: Algebraic Geometry, Group Theory and Generalizations, Ordinary Differential Equations

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