Discrete Relations On Abstract Simplicial Complexes

Talk at CASC 2005, Kalamata, Greece

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13 September 2005

Outline

What We Call a System Of Discrete Relations?

System of discrete relations on abstract simplicial complex is

1. generalization of cellular automaton

2. set-theoretic analog of system of polynomial equations

System Of Discrete Relations Vs. Cellular Automaton

System of discrete relations	Cellular automaton
Space–time arbitrary abstract simplicial complex	**Space–time** regular uniform lattice
describes models with dynamically emerging and evolving space-time	describes only preexisting and fixed space-time
Local law set of relations imposed on states of points of simplices in complex	**Local law** local site-update function (i.e. functional relation)
much more flexible setting of laws	too restrictive in applications; violates symmetry of points

Discrete Relations As Analog Of Polynomial Equations

Let $\{x_1, \ldots, x_k\}$ be a set of points taking values in a q-element set of states S. Canonically $S = \{0, \ldots, q-1\}$.

If number of states $|S| = q = p^n$,
the set S can be equipped with the structure of Galois field $\mathbb{F}_q$.

The functional completeness of polynomials over $\mathbb{F}_q$ implies that if $q = p^n$, then arbitrary k-ary relation $R^k \subseteq S^k$ can be realized by a set of zeros of polynomial $f(x_1, \ldots x_k) \in \mathbb{F}_q[x_1, \ldots x_k]$.

- This analogy is useful but not necessary (and fails if $q \neq p^n$).
- The best way to construct polynomial from relation is multivariate Lagrange interpolation.

Abstract Simplicial Complex $K = (X, \Delta)$

1. $X = \{x_0, x_1, \ldots\}$ is a finite (or countable) set of points.
2. Δ is a collection of subsets of X called simplices such that
   1. for all $x_i \in X$, $\quad \{x_i\} \in \Delta$
   2. if $\tau \subseteq \delta \in \Delta$, then $\tau \in \Delta$

2.2 $\Longrightarrow$ Δ is uniquely defined by maximal simplices under inclusion.

Dimension of simplex δ : $\dim \delta = |\delta| - 1$
Motivation: $k + 1$ points generically embedded in Euclidean space $\mathbb{R}^{n \geq k}$ form k-dimensional geometric simplex

Dimension of complex K : $\dim K = \max_{\delta \in \Delta} \dim \delta$
Nöbeling–Pontryagin theorem $\Longrightarrow$ any k-dimensional abstract complex realizable geometrically in $\mathbb{R}^{2k+1}$

The Holographic Principle By G. ’t Hooft

also J.D. Bekenstein, L. Susskind, L. Smolin and many others

Combination of quantum mechanics and gravity implies:

1. The world at the Planck scale can be described by a 3D discrete lattice theory with a spacing of order the Planck length.

2. Events on the 3D lattice are determined by a set of Boolean data on a 2D lattice at the spatial boundaries of the world.

3. Transfer of data from 2 to 3 dimensions is controlled by some physically motivated local relations on plaquettes of the lattice.

Relation Over a Set Of Points

A relation is generally defined as a subset of a Cartesian product $S \times \cdots \times S$ of the set of states.

The notation $S^{\{x_i\}}$ specifies the set S as a set of values for the point x_i.

For the k-set $\delta = \{x_1, \ldots, x_k\}$ we denote $S^\delta \equiv S^{\{x_1\}} \times \cdots \times S^{\{x_k\}}$.

Definition (Relation)

A relation R^δ over a set $\delta = \{x_1, \ldots, x_k\}$ is any subset of the hypercube S^δ, i.e., $R^\delta \subseteq S^\delta$.

Extension Of Relation To a Larger Set Of Points

Definition (Extension of relation)

Given a set of points δ, its subset $\tau \subseteq \delta$ and relation R^{τ} over the subset τ, we define extension of R^{τ} as the relation

$$R^{\delta} = R^{\tau} \times S^{\delta \setminus \tau}$$

The procedure of extension allows to extend relations $R^{\delta_1}, \ldots, R^{\delta_m}$ defined on different domains to the common domain, i.e., the union $\delta_1 \cup \cdots \cup \delta_m$.

Consequences Of Relation

Definition (Consequence of relation)

A relation Q^δ is a consequence of relation R^δ if $R^\delta \subseteq Q^\delta \subseteq S^\delta$, i.e., Q^δ is any superset of R^δ.

The total number of all consequences (including R^δ itself and the trivial relation S^δ) is, obviously,

$$2^{\left(q^k - |R^\delta|\right)}.$$

It is natural to distinguish consequences reduced to relations over smaller sets of points:

Definition (Proper consequence)

A nontrivial relation Q^τ is called proper consequence of relation R^δ if τ is a proper subset of δ, i.e., $\tau \subset \delta$, and relation $Q^\tau \times S^{\delta\setminus\tau}$ is consequence of R^δ.

Prime Relations

There are relations without proper consequences and these relations are most fundamental for a given number of points k.

Definition (Prime relation)

A relation without proper consequences is called prime.

Compatibility Condition Of a Set Of Relations $R^{\delta_1}, \ldots, R^{\delta_m}$

Naturally this is intersection of extensions of the relations to the common domain:

Definition (Base relation ≡ Compatibility condition)

A base relation of the system of relations $R^{\delta_1}, \ldots, R^{\delta_m}$ is relation

$$R^{\delta} = \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta \setminus \delta_i} \right), \text{ where } \delta = \bigcup_{i=1}^{m} \delta_i.$$

Remarks on the polynomial case $q = p^n$:

1. The compatibility condition can be represented by a single polynomial, in contrast to the Gröbner basis approach.
2. Any possible Gröbner basis of polynomials representing $R^{\delta_1}, \ldots, R^{\delta_m}$ corresponds to some combination of consequences of the base relation.

Decomposition Of Relation With Proper Consequences

Definition (Canonical decomposition)

A canonical decomposition of the relation R^{δ} with proper consequences $R^{\delta_1}, \ldots, R^{\delta_m}$ is the representation

$$R^{\delta} = PR^{\delta} \bigcap \left(\bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta \setminus \delta_i} \right) \right)$$

Definition (Principal factor)

A pricipal factor of relation R^{δ} with proper consequences $R^{\delta_1}, \ldots, R^{\delta_m}$ is relation

$$PR^{\delta} = R^{\delta} \bigcup \left(S^{\delta} \setminus \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta \setminus \delta_i} \right) \right)$$

Principal factor is relation of maximum “freedom” which must be added to consequences in order to restore R^{δ}.

Relation Completely Determined By Relations Over Smaller Sets Of Points

If principal factor in the canonical decomposition is trivial, the relation is reducible to its proper consequences:

Definition (Reducible relation)

We call a relation R^δ reducible if it can be represented in the form

$$R^\delta = \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta \setminus \delta_i} \right), \quad \delta_i \text{ are proper subsets of } \delta.$$

How To Set Topology On Arbitrary Relation?

Imposing structure of abstract simplicial complex, homology, cohomology etc.

For any n-ary relation $R \subseteq S^n = \underbrace{S \times \cdots \times S}_{n}$

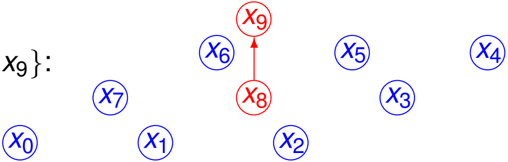
we can construct abstract simplicial complex with all accompanying things like chain and cochain complexes, homology group, cohomology ring etc.

defining correspondence:

n dimensions of hypercube S^n	n points $X = \{x_1, \ldots, x_n\}$
Domains of irreducible components of R	Maximal simplices determining set of simplices Δ

Local Rule Of ***Life***

State set: $S = \{0, 1\}$
Domain is 10-set $\delta = \{x_0, \ldots, x_9\}$:



Relation $R^{\delta}_{\mathbf{Life}}$ $\left(\left|R^{\delta}_{\mathbf{Life}}\right| = 512\right)$ in logical form

$$\text{either} \quad \sum_{i=0}^{7} x_i = 3 \wedge x_9 = 1 \vee \sum_{i=0}^{7} x_i = 2 \wedge x_8 = x_9,$$

$$\text{or} \qquad x_9 = 0 \quad \text{otherwise}$$

in polynomial form over $\mathbb{F}_2$

$$P_{\mathbf{Life}} = x_9 + x_8 \{\sigma_7 + \sigma_6 + \sigma_3 + \sigma_2\} + \sigma_7 + \sigma_3 = 0,$$

$\sigma_k \equiv \sigma_k(x_0, \ldots, x_7)$ is kth elementary symmetric polynomial

Local Relation $R^{\delta}_{\boldsymbol{Life}}$ Is Reducible

$$R^{\delta}_{\boldsymbol{Life}} = R_2^{\delta\setminus\{x_8\}} \bigcap \left(\bigcap_{k=0}^{6} R_1^{\delta\setminus\{x_{i_k}\}} \right) \quad \left(\text{trivial factors } S^{\{x_i\}} \text{omitted}\right)$$

Eight relations $R_1^{\delta\setminus\{x_i\}}$ $(0 \leq i \leq 7$, any 7 of them enough$)$ in polynomial form:

$$x_8 x_9 \left\{\sigma^i_6 + \sigma^i_5 + \sigma^i_2 + \sigma^i_1\right\} + x_9 \left\{\sigma^i_6 + \sigma^i_2 + 1\right\} + x_8 \left\{\sigma^i_7 + \sigma^i_6 + \sigma^i_3 + \sigma^i_2\right\} = 0,$$

$$\sigma^i_k \equiv \sigma_k\left(x_0, \ldots, \widehat{x_i}, \ldots, x_7\right).$$

Polynomial form of $R_2^{\delta\setminus\{x_8\}}$:

$$x_9 \left\{\sigma_7 + \sigma_6 + \sigma_3 + \sigma_2 + 1\right\} + \sigma_7 + \sigma_3 = 0.$$

Relations $R_1^{\delta\setminus\{x_i\}}$ and $R_2^{\delta\setminus\{x_8\}}$ are irreducible but not prime,
continue decompositions ...

Final Decomposition Of $R^{\delta}_{\boldsymbol{Life}}$ In Polynomials

$$
\begin{aligned}
x_8 x_9 \left\{\sigma_2^i + \sigma_1^i\right\} + x_9 \left\{\sigma_2^i + 1\right\} + x_8 \left\{\sigma_7^i + \sigma_6^i + \sigma_3^i + \sigma_2^i\right\} &= 0, \\
x_9 \left\{\sigma_3 + \sigma_2 + 1\right\} + \sigma_7 + \sigma_3 &= 0, \\
(x_8 x_9 + x_9) \left\{\sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1\right\} &= 0, \\
x_9 \left\{\sigma_3^i + \sigma_2^i + \sigma_1^i + 1\right\} &= 0, \\
x_9 x_{i_0} x_{i_1} x_{i_2} x_{i_3} &= 0, \\
\sigma_k^{ij} \equiv \sigma_k \left(x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_7\right).
\end{aligned}
$$

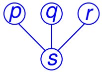
All the above takes “zero” time ($<$ 1 sec) on a 1.8GHz AMD Athlon notebook with 960Mb.

Gröbner basis computations over $\mathbb{F}_2$ with Maple 9 in most term orderings give the same set of polynomials (modulo several symmetry violating reductions) for times from 33 min to 1 h 22 min.

There Are 256 Elementary Cellular Automata
118 reducible, 138 irreducible (including 2 prime)

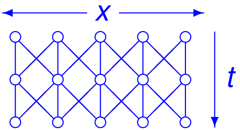
State set: $S = \{0, 1\}$

Domain of local rule is 4-set $\delta = \{p, q, r, s\}$:



Local rule is functional relation: $s = f(p, q, r)$

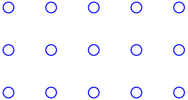
Space-time is lattice with integer coordinates (x, t) :



$x \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ or $x \in \mathbb{Z}_m$ (spatial m-periodicity)
$t \in \mathbb{Z}^* = \{0, 1, \ldots\}$
$u(x, t) \in S = \{0, 1\}$ denotes a state of point on the lattice.

Two Trivial Automata

Automata 0 and 255 in Wolfram's numeration are defined on disjoint union of vertices



Polynomial forms of relations: $\left\{ \begin{array}{lll} s = 0 & \text{rule} & 0 \\ s + 1 = 0 & \text{rule} & 255 \end{array} \right.$

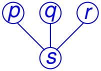
Remark: unary relations are called properties

Six Automata 15, 51, 85, 170, 204, 240

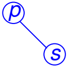
are disjoint collections of (spatially) zero-dimensional automata

For example automaton 15:

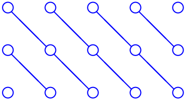
Local relation with bit table (characteristic function)

0101010110101010 on the simplex 

is reduced

to 0110 on the face 

and the lattice splits into



Automaton 15 is “integrable”

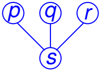
Bit table 0110 gives general solution:

$$u(x,t) = u(x-t,0) + t \mod 2$$

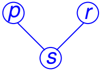
Ten Automata 5, 10, 80, 90, 95, 160,165,175,245, 250

are decomposed into two identical automata

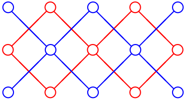
Example: automaton 90 producing fractal
(of dimensions: topological 1, Hausdorff $\ln 3/\ln 2 \approx 1.58$)
known as Sierpinski sieve (gasket, triangle)
Local relation with bit table

1010010101011010 on the simplex 

is reduced

to 10010110 on the face 

The lattice splits into two independent complexes

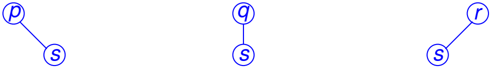


General solution of automaton 90:

$$u(x,t)=\sum_{k=0}^{t}\binom{t}{k}u(x-t+2k,0) \mod 2$$

Proper Consequence May Give Valuable Information in the case of irreducible relation also

E. g., there are 64 automata having non-functional proper consequences with bit table 1101 on at least one of faces

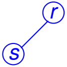


Polynomial forms: $ps + s = 0$ $qs + s = 0$ $rs + s = 0$
Relation 1101 imposes severe restriction on automaton's dynamics: after 1st 0 only 0's can be along corresponding diagonal or vertical, since for, say, $ps + s = 0$:

$$p = 1 \Longrightarrow s = 0 \vee s = 1 \quad \text{but} \quad p = 0 \Longrightarrow s = 0$$

Picture From Wolfram's Atlas http://atlas.wolfram.com/01/01/

Local relation $pqr + qr + pr + s = 0$ of automaton 168 has proper consequence $rs + s = 0$ on face



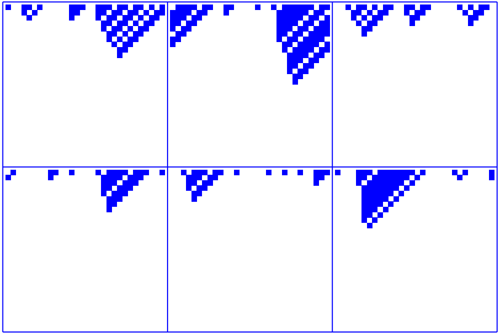


Figure: Rule 168. Several random initial conditions

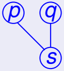
Automaton 30 With Chaotic Behavior

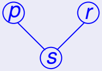
used as random number generator in ***Mathematica***

Relation: 1001010101101010 or $qr + s + r + q + p = 0$

Proper consequences:

face





bit table	11011110	11011110
polynomial	$qs + pq + q$	$rs + pr + r$

Principal factor: 1011111101111111 or

$$qrs + pqr + rs + qs + pr + pq + s + p = 0$$

Gröbner basis in degree-reverse-lexicographic order:

$$\{qr + s + r + q + p,\ qs + pq + q,\ rs + pr + r\}$$

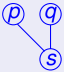
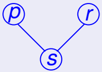
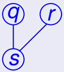
Here our polynomials coincide with polynomials from Gröbner basis

Universal Automaton 110

is capable (like a Turing machine) of simulating any computational process

Relation: 1100000100111110 or $pqr + qr + s + r + q = 0$

Proper consequences:

face			
bit table	11011111	11011111	10010111
polynomial	$pqs + qs + pq + q$	$prs + rs + pr + r$	$qrs + s + r + q$

Principal factor: 1111111111111110 or $pqrs = 0$

Gröbner basis in degree-reverse-lexicographic order:

$$\{prs + rs + pr + r,\ qs + rs + r + q,\ qr + rs + s + q,\ pr + pq + ps\}$$

Gröbner basis contains different set of polynomials

Summary

- System of discrete relations on abstract simplicial complex

  Can be interpreted as
  - natural generalization of cellular automaton
  - set-theoretic analog of system of polynomial equations
- Algorithms for
  - compatibility analysis of system of discrete relations
  - canonical decompositions of discrete relations
- Topology on arbitrary discrete relation

  Canonical decomposition implies

  structure of abstract simplicial complex:

dimensions of relation	$\Leftrightarrow$	points (vertices)
domains of irreducible components	$\Leftrightarrow$	maximal simplices
- Cellular automata

  presence of (nonfunctional) proper consequences

  may determine global behavior of automaton