Application of Finite Groups to Quantum Physics

Polynomial Computer Algebra PCA '2011
St. Petersburg, Russia, April 17–22, 2011

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April 20, 2011

Preliminaries

1. Quantum behavior is manifestation of purely mathematical properties of systems with indistinguishable objects — any violation of identity of quantum particles destroys quantum interferences
2. For systems with symmetries only invariant (independent of relabeling of "homogeneous" elements) relations and statements are objective. E.g., no objective meaning can be attached to electric potentials φ and ψ or to space points $\mathbf{a}$ and $\mathbf{b}$, but invariants $\psi - \varphi$ and $\mathbf{b} - \mathbf{a}$ (in more general group notation $\varphi^{-1}\psi$ and $\mathbf{a}^{-1}\mathbf{b}$) are meaningful
3. Question of "whether the real world is discrete or continuous" or even "finite or infinite" is metaphysical — neither empirical observations nor logical arguments can validate one of the two adoptions. The choice is a matter of belief or taste

Since no empirical consequences of choice between finite (discrete) and infinite (continuous) descriptions are possible — "physics is independent of metaphysics" — we can consider quantum concepts in constructive finite background without any risk to destroy physical content of problem

Classical and Quantum Evolution of Dynamical System

Classical evolution is a sequence of states evolving in time

$\cdots \to s_{t-1} \to s_t \to s_{t+1} \to \cdots \qquad t \in \mathcal{T} \subseteq \mathbb{Z} \qquad S = \{s_1, \ldots, s_N\}$

Quantum evolution is a sequence of permutations of states

$\cdots \to p_{t-1} \to p_t \to p_{t+1} \to \cdots \qquad p_t \in G = \{g_1, \ldots, g_M\} \leq \mathrm{Sym}(S)$

In physics systems with space $X = \{x_1, \ldots, x_{|X|}\}$ are important

- Set of states takes special structure of functions on space $S = \Sigma^X$
- $\Sigma = \{\sigma_1, \ldots, \sigma_{|\Sigma|}\}$ is set of local states
- Space symmetry group $F = \{f_1, \ldots, f_{|F|}\} \leq \mathrm{Sym}(X)$
- Internal symmetry group $\Gamma = \{\gamma_1, \ldots, \gamma_{|\Gamma|}\} \leq \mathrm{Sym}(\Sigma)$
- Whole symmetry group G can be expressed as split extension $\mathbf{1} \to \Gamma^X \to G \to F \to \mathbf{1}$ in terms of space and internal symmetries

Interpreting Quantum Amplitude as Gauge Connection

Feynman's path amplitude is product of group elements

$$A_{\mathrm{U}(1)} = A_0 \exp(iS) = A_0 \exp\left(i \int_0^T L dt\right) \to A_0 \mathrm{e}^{iL_{0,1}} \ldots \mathrm{e}^{iL_{t-1,t}} \ldots \mathrm{e}^{iL_{T-1,T}}$$

Let us interpret $\rho(t-1,t) = \mathrm{e}^{iL_{t-1,t}}$ as $\mathrm{U}(1)$-connection

$\mathrm{U}(1)$ is 1D unitary representation of circle $\Gamma = S^1 \equiv \mathbb{R}/\mathbb{Z}$ – Lie group

Straightforward generalization

$$A_{\rho(\Gamma)} = \rho\left(\alpha_{T,T-1}\right) \ldots \rho\left(\alpha_{t,t-1}\right) \ldots \rho\left(\alpha_{1,0}\right) A_0 \qquad \alpha_{t,t-1} \in \Gamma$$

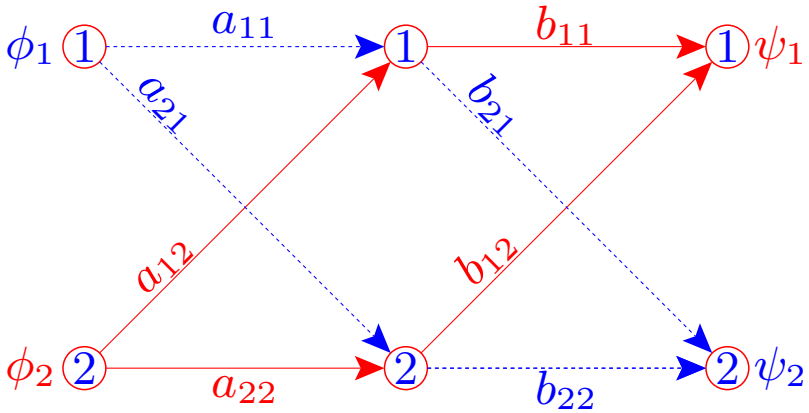
- Γ is some group, not necessarily circle
- ρ is some unitary representation of Γ, not necessarily 1-dimensional

We assume Γ is finite group

Unifying Feynman's and Matrix Formulations of Quantum Mechanics

Feynman's rules "multiply subsequent events" and "sum up alternative histories" is nothing else than rephrasing of matrix multiplication rules

Quantum evolution $\quad |\psi\rangle = U\,|\phi\rangle \qquad |\phi\rangle = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$



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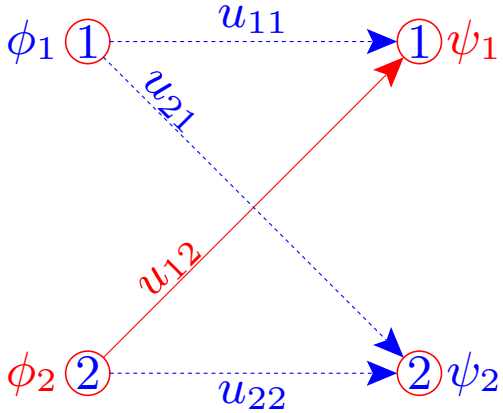


$$\Updownarrow \qquad\qquad \Updownarrow$$

$$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} \sim U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$BA = U$$

All this works also for generalized amplitude with non $\mathrm{U}(1)$-valued connection assuming non-commutativity of matrix entries

Sketch of Quantum Mechanics

1. Quantum description deals with unitary operators U acting in Hilbert spaces $\mathcal{H}$ whose elements $|\psi\rangle \in \mathcal{H}$ are called "states", "state vectors", "wave functions", "amplitudes" etc.
2. Quantum mechanical particles are associated with unitary representations of some groups. The representations are called "singlets", "doublets" etc. in accordance with their dimensions. Multidimensional representations describe spin
3. Quantum mechanical evolution is unitary transformation of initial state vector $|\psi_{in}\rangle$ into final $|\psi_{out}\rangle = U\,|\psi_{in}\rangle$
4. Quantum mechanical experiment (observation, "measurement") is comparison of state $|\psi\rangle$ of system with state $|\phi\rangle$ of apparatus
5. Born's rule: probability to register a particle described by $|\psi\rangle$ by apparatus tuned to $|\phi\rangle$ is equal to $\dfrac{|\langle\phi \mid \psi\rangle|^2}{\langle\phi \mid \phi\rangle\,\langle\psi \mid \psi\rangle}$

Our goal is to implement all this in finite background

Permutations and Linear Representations

- Any set $\Omega = \{\omega_1, \ldots, \omega_n\}$ with transitive symmetries $\mathrm{G} = \{g_1, g_2, \ldots, g_M\}$ is in 1-to-1 correspondence with right (or left) cosets of some subgroup $H \leq \mathrm{G}$

  $\Omega \cong H \backslash \mathrm{G}$ (or G/H) is called homogeneous space (G-space)

  Action of G on Ω is faithful if H does not contain normal subgroups of G

  Action by permutations $\pi(g) = \begin{pmatrix} \omega_i \\ \omega_i g \end{pmatrix} \cong \begin{pmatrix} Ha \\ Hag \end{pmatrix} \quad g, a \in \mathrm{G} \quad i = 1, \ldots, \mathrm{n}$

- Maximum transitive set $\Omega \cong \{\mathbf{1}\} \backslash \mathrm{G} \cong \mathrm{G}$ corresponds to right regular action

$$\Pi(g) = \begin{pmatrix} \mathrm{g}_i \\ \mathrm{g}_i g \end{pmatrix} \quad i = 1, \ldots, \mathrm{M}$$

- For "quantitative" ("statistical") description elements of Ω are equipped with numerical "weights" from suitable number system $\mathcal{N}$ containing 0 and 1 — permutations can be rewritten as matrices

  $\pi(g) \to \rho(g) \quad \rho(g)_{ij} = \delta_{\omega_i g, \omega_j} \quad i, j = 1, \ldots, \mathrm{n}$ permutation representation

  $\Pi(g) \to \mathrm{P}(g) \quad \mathrm{P}(g)_{ij} = \delta_{e_i g, e_j} \quad i, j = 1, \ldots, \mathrm{M}$ regular representation

- For the sake of freedom of algebraic manipulations, one assumes usually that $\mathcal{N}$ is algebraically closed field — normally complex numbers $\mathbb{C}$.
  If $\mathcal{N}$ is a field, then the set Ω can be treated as basis of linear vector space $\mathcal{H} = \mathrm{Span}\,(\omega_1, \cdots, \omega_n)$.

Constructive Number Systems

The field $\mathbb{C}$ consists almost entirely of useless non-constructive elements
What is needed actually are combinations of basics:

- Natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ — different counters
- Irrationalities (all are cyclotomic integers):
  - Roots of unity — eigenvalues of linear representations
  - Square roots of dimensions of representations — normalizing coefficients to provide unitarity

Derivatives:

- Ring $\mathbb{Z}$ can be introduced via definition $(-1) = \sum_{k=1}^{p-1} \mathrm{r}^{\frac{E}{p}k}$

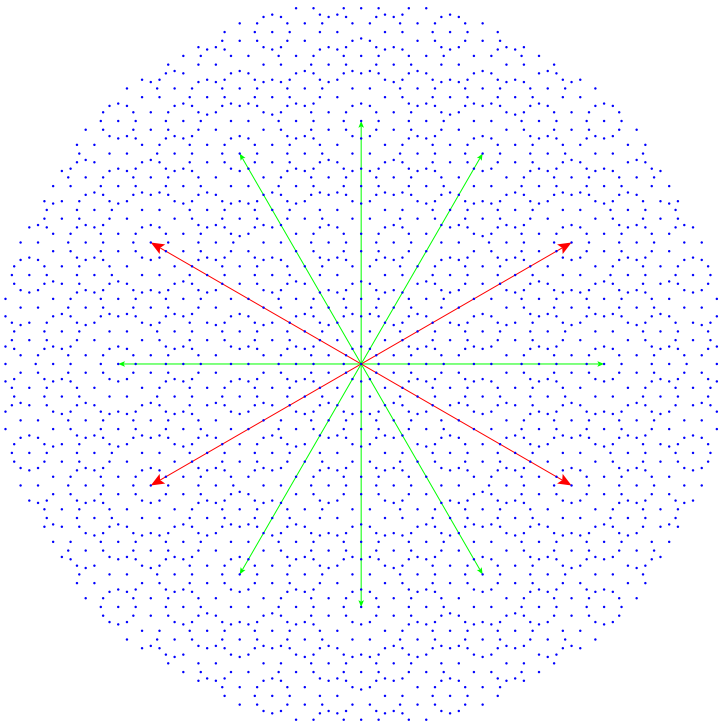
  p is any factor of group exponent E, r is primitive E^{th} root of unity
- Ring of cyclotomic integers $\mathcal{N}_{\mathcal{P}} = \mathbb{Z}[\mathrm{r}] / \langle \Phi_{\mathcal{P}}(\mathrm{r}) \rangle$
  $\Phi_{\mathcal{P}}(\mathrm{r})$ is $\mathcal{P}^{\text{th}}$ cyclotomic polynomial
- $\mathcal{P}^{\text{th}}$ cyclotomic field $\mathbb{Q}_{\mathcal{P}} = \mathbb{Q}[\mathrm{r}] / \langle \Phi_{\mathcal{P}}(\mathrm{r}) \rangle$ — in fact, quotient field of $\mathcal{N}_{\mathcal{P}}$
- Minimal abelian number field $\mathcal{F} \leq \mathbb{Q}_{\mathcal{P}}$ containing given irrationalities

All irrationalities are intermediate elements of quantum description

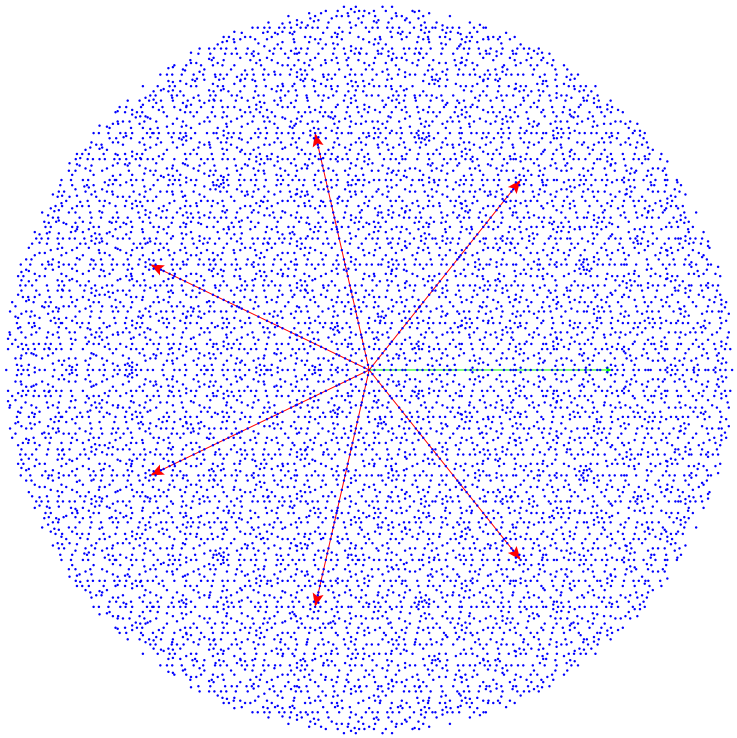
disappearing in final expressions for quantum observables

Embedding Cyclotomic Integers $\mathcal{N}_{\mathcal{P}}$ into Complex Plane $\mathbb{C}$

$\mathcal{P} = 12$



$\mathcal{P} = 7$



Red (green) arrows — primitive (nonprimitive) roots

Complex conjugation in $\mathcal{N}_{\mathcal{P}}$ is defined via rule $\overline{\mathrm{r}^k} = \mathrm{r}^{\mathcal{P}-k}$

Linear Representations of Finite Group

1. Any linear representation of G is unitary — there is always unique invariant inner product $\langle \cdot \mid \cdot \rangle$ making $\mathcal{H}$ into Hilbert space
2. All possible irreducible unitary representations of G are contained in regular representation

$$\mathrm{T}^{-1}\mathrm{P}(g)\mathrm{T} = \begin{pmatrix} D_1(g) & & & & & \\ & \left.\begin{matrix} D_2(g) & & \\ & \ddots & \\ & & D_2(g) \end{matrix}\right\} d_2 & & & & \\ & & \ddots & & & \\ & & & \left.\begin{matrix} D_m(g) & & \\ & \ddots & \\ & & D_m(g) \end{matrix}\right\} d_m & \\ & & & & \end{pmatrix}$$

- m = number of $\begin{cases} \text{different irreducible representations } D_j \text{ of } \mathrm{G} \\ \text{conjugacy classes in } \mathrm{G} \end{cases}$
- $d_j = \dim D_j$ = multiplicity of D_j in regular representation
- obviously $d_1^2 + d_2^2 + \cdots + d_m^2 = \mathrm{M} \equiv |\mathrm{G}|$ besides: d_j divides M

Finite Quantum System

1. Some abelian number field $\mathcal{F}$ instead of complex numbers $\mathbb{C}$
2. States $|\psi\rangle$ form K-dimensional Hilbert space $\mathcal{H}_{\mathrm{K}}$ over $\mathcal{F}$
   The Born rule is formulated as usual via inner product in $\mathcal{H}_{\mathrm{K}}$
3. Operators of evolution $|\psi_{out}\rangle = U\,|\psi_{in}\rangle$ are elements of unitary representation U of finite group: $U = \mathrm{U}(g)\,,\ \ g \in \mathrm{G}$

Comments:

- Clearly only finite number of evolutions is possible at all

$$U \in \{\mathrm{U}(\mathrm{g}_1)\,, \mathrm{U}(\mathrm{g}_2)\,, \ldots, \mathrm{U}(\mathrm{g}_{\mathrm{M}})\}$$

- Hermitian operator called Hamiltonian or energy operator can be introduced $U = \mathrm{e}^{-iH} \Longrightarrow H = i \ln U \equiv \sum_{k=0}^{p-1} \lambda_k U^k$, p is period of U
- More generally, hermitian operators describing observables in quantum formalism can be expressed in terms of group algebra representation:

$$A = \sum_{k=1}^{\mathrm{M}} \alpha_k \mathrm{U}(\mathrm{g}_k)$$

Reducing Quantum Problem to Permutations

Any irreducible representation of G is contained in regular one

$$\Downarrow$$

Any unitary representation U in K-dimensional Hilbert space $\mathcal{H}_{\mathrm{K}}$ can be extended to N-dimensional ($\mathrm{N} \geq \mathrm{K}$) permutation representation $\widetilde{\mathrm{U}}$ in Hilbert space $\mathcal{H}_{\mathrm{N}}$

Comments:

- Thus any quantum problem in K-dimensional Hilbert space can be reduced to permutations of N things
- Case $\mathrm{N} > \mathrm{K}$ (strictly greater) is most interesting
- Additional "hidden parameters" — appearing due to increase of number of states (dimension of space) — in no way can affect data relating to space $\mathcal{H}_{\mathrm{K}}$ since both $\mathcal{H}_{\mathrm{K}}$ and its complement in $\mathcal{H}_{\mathrm{N}}$ are invariant subspaces of extended space $\mathcal{H}_{\mathrm{N}}$

Born's Rule Connects Mathematical Description with Observation

- Unitary inner product $\langle \cdot \mid \cdot \rangle$ can always be constructed from arbitrary $(\cdot \mid \cdot)$ by "averaging over group"

$$\langle \phi \mid \psi \rangle = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} (\mathsf{U}(g)\phi \mid \mathsf{U}(g)\psi)$$

- "Probability" to observe system in state ψ by apparatus tuned to ϕ is determined by the Born rule

$$\mathbf{P}(\phi, \psi) = \frac{|\langle \phi \mid \psi \rangle|^2}{\langle \phi \mid \phi \rangle \langle \psi \mid \psi \rangle}$$

or in more symmetric with respect to the pair "system–apparatus" way

$$\mathbf{P}(\phi, \psi) = \frac{|\langle \phi \mid \psi \rangle|^2}{|\langle \phi \mid \psi \rangle|^2 + \|\phi \wedge \psi\|^2}$$

Comments:

- Some elements of description of systems with symmetries are unobservable since there is no way to distinguish objects lying on the same group orbit

  — only invariant combinations of elements of description are observable

- State vectors ϕ and ψ are unobservable
- Born's probability $\mathbf{P}(\phi, \psi)$ is observable invariant of representation of G

Born's Probability in Permutation Representation and in Invariant Subspaces

- Consider group $G = \{g_1, \ldots, g_M\}$ permuting elements of set $S = \{s_1, \ldots, s_N\}$
- System and apparatus state vectors in permutation representation

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} \quad \text{and} \quad |m\rangle = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix}$$

Natural assumption: n_i and m_i are (unobservable) natural numbers

— "multiplicities of occurrences" of $s_i \in S$ in system and apparatus states

- Born's probability for permutations

$$\mathbf{P}(m, n) = \frac{\left(\sum_{i=1}^{N} m_i n_i\right)^2}{\sum_{i=1}^{N} m_i^2 \sum_{i=1}^{N} n_i^2}$$

is rational number $> 0 \implies$ destructive interference is impossible

However, destructive interference of vectors with natural components can be observed in proper invariant subspaces of permutation representation

Quantum behavior is manifestation of invariant properties of a system visible in invariant subspaces of permutation representation of symmetry group of the system

Example: Group of Permutations of Three Things S_3

Faithful action on $\Omega = S_2 \backslash S_3 = \{1, 2, 3\}$

$$G = S_3 = \{\overbrace{g_1 = ()}^{K_1}, \overbrace{g_2 = (23), g_3 = (13), g_4 = (12)}^{K_2}, \overbrace{g_5 = (123), g_6 = (132)}^{K_3}\}$$

S_3 can be generated by two generators g_2 and g_6 (one of many possible choices)

Permutation matrices of generators $P_2 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}, \quad P_6 = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}$

Class multiplication table — its coefficients (natural numbers) are called class coefficients

$$K_1 K_j = K_j, \quad K_2^2 = 3K_1 + 3K_3, \quad K_2 K_3 = 2K_2, \quad K_3^2 = 2K_1 + K_3$$

Character table

	K_1	K_2	K_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

3D permutation representation P is equivalent to sum $\widetilde{U} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$ of 2D faithful (character χ_3) U and 1D trivial 1 (character χ_1) representations: $\widetilde{U} = \mathrm{T}^{-1} P \mathrm{T}$

S_3. Constructing "Quantum Basis" via Diagonalizing P_6 Leads to Monomial Representation

Matrices of 2D faithful representation for generators

$$U_2 = \begin{pmatrix} 0 & r^2 \\ r & 0 \end{pmatrix}, \quad U_6 = \begin{pmatrix} r & 0 \\ 0 & r^2 \end{pmatrix}, \quad \text{r is primitive 3d root of unity}$$

Transformation matrix

$$\mathrm{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & r^2 \\ 1 & r^2 & 1 \\ 1 & r & r \end{pmatrix}, \quad \mathrm{T}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & r & r^2 \\ r & 1 & r^2 \end{pmatrix}$$

S_3. Diagonalizing P_2 Leads to Representation Used in Particle Physics

Matrices of 2D faithful representation for generators

$$U_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_6' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Transformation matrix

$$\mathrm{T}' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathrm{T}'^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is called Harrison-Perkins-Scott or tribimaximal mixing matrix.

It is used to description of neutrino oscillation data

S_3. Projecting States into Invariant 2D Subspace

- Recall $\widetilde{U} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$, $\widetilde{U} = \mathrm{T}^{-1} P \mathrm{T}$, $\mathrm{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mathrm{r}^2 \\ 1 & \mathrm{r}^2 & 1 \\ 1 & \mathrm{r} & \mathrm{r} \end{pmatrix}$
- State vectors in:
  - "permutation basis" $|n\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, $|m\rangle = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$
  - "quantum basis"

$$\left|\widetilde{\psi}\right\rangle = \mathrm{T}^{-1} |n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2 + n_3 \\ n_1 + n_2 \mathrm{r} + n_3 \mathrm{r}^2 \\ n_1 \mathrm{r} + n_2 + n_3 \mathrm{r}^2 \end{pmatrix}, \quad \left|\widetilde{\phi}\right\rangle = \mathrm{T}^{-1} |m\rangle = \cdots$$

- Projections onto U: $|\psi\rangle = \begin{pmatrix} n_1 + n_2 \mathrm{r} + n_3 \mathrm{r}^2 \\ n_1 \mathrm{r} + n_2 + n_3 \mathrm{r}^2 \end{pmatrix}$, $|\phi\rangle = \cdots$

S_3. Quantum Interference in Invariant Subspace

- Born's probability for $2D$ state vectors in terms of $3D$ parameters

$$\mathbf{P}(\phi,\psi) = \frac{|\langle\phi \mid \psi\rangle|^2}{\langle\phi \mid \phi\rangle\langle\psi \mid \psi\rangle} = \frac{\left(3Q_P(m,n) - L_P(m)\,L_P(n)\right)^2}{\left(3Q_P(m,m) - L_P(m)^2\right)\left(3Q_P(n,n) - L_P(n)^2\right)}$$

- $L_P(n) = n_1 + n_2 + n_3$ and $Q_P(m,n) = m_1 n_1 + m_2 n_2 + m_3 n_3$ are linear and quadratic invariants of permutation representation P
- Condition for destructive quantum interference

$$3\left(m_1 n_1 + m_2 n_2 + m_3 n_3\right) - \left(m_1 + m_2 + m_3\right)\left(n_1 + n_2 + n_3\right) = 0$$

has infinitely many solutions in natural numbers, e.g., $|n\rangle = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $|m\rangle = \begin{pmatrix}1\\3\\2\end{pmatrix}$

Thus, we obtained essential features of quantum behavior

from "permutation dynamics" and "natural" interpretation of quantum amplitude by simple transition to invariant subspace

State Mixing in Flavor Physics

Fermions in Standard Model form 3 generations of quarks and leptons

Fermions \ Generations	1	2	3
Up-quarks	u	c	t
Down-quarks	d	s	b
Charged leptons	e^-	μ^-	τ^-
Neutrinos	ν_e	ν_μ	ν_τ

Transitions between up- and down- quarks in quark sector and flavor and mass neutrino states in lepton sector are described by Cabibbo–Kobayashi–Maskawa

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

and Pontecorvo–Maki–Nakagawa–Sakata

$$U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

mixing matrices

Observational Evidences of Fundamental Finite Symmetries

Most sharp picture comes from numerous neutrino oscillation data

Phenomenological pattern

- ν_μ and ν_τ flavors are presented with equal weights in all 3 mass eigenstates ν_1, ν_2, ν_3 (called "bi-maximal mixing")
- all three flavors are presented equally in ν_2 ("trimaximal mixing")
- ν_e is absent in ν_3

implies probabilities $\left(|U_{\alpha\beta}|^2\right) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

$\longrightarrow$ unitary matrix (Harrison, Perkins, Scott) $U_{\text{HPS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

U_{HPS} (also "tribimaximal mixing matrix") coincides — up to renaming $\nu_1 \rightleftarrows \nu_2$ — with matrix T' decomposing permutation representation of S_3 into irreducible components

This caused a burst of activity in building models based on finite symmetry groups

In the quark sector the picture is not so clear, but there are some encouraging empirical observations, e.g., quark-lepton complementarity (QLC)

Conclusions

1. Quantum mechanics is merely *a priori* mathematical scheme based on fundamental impossibility to trace identity of homogeneous objects in their evolution — some kind of "*calculus of indistinguishables*"
2. Only *"statistical"* statements about numbers of certain invariant combinations of elements may have objective significance
3. These objective statements can be expressed in terms of group invariants and natural numbers attached to symmetry groups: dimensions of irreducible representations, class coefficients, etc.
4. There are observational evidence of finite symmetries in fundamental physical processes. If we adopt idea that fundamental symmetries are *per se* finite rather than remnants of continuous symmetries, then continuous unitary groups used in physical theories can be treated simply as repositories for all finite groups having faithful representations of respective dimension