Towards an algorithmisation of the Dirac constraint formalism

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Contents

Contents

Degenerate Lagrangian Systems

Modern theories of gravity and elementary particle physics contain gauge degrees of freedom and by this reason are described by degenerate Lagrangians.

In mechanics: Lagrangian $L(q, \dot{q})$ is a function of (generalized) coordinates $q := q_1, q_2, \ldots, q_n$ and velocities $\dot{q} := \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.

The Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,, \qquad 1 \leq i \leq n$$

have the structure

$$H_{ij}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\,\dot{q}_j - \frac{\partial L}{\partial q_i} = 0\,, \qquad H_{ij} := \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

Degenerate Lagrangian Systems

Lagrangian $L(q, \dot{q})$ is

1. regular if $r := rank\|H_{ij}\| = n$
2. degenerate (singular) if $r < n$

In the 1st case the Euler-Lagrange equations are solved with respect to the accelerations ($\ddot{q}$), and there is no hidden constraints.

In the 2nd case the equations cannot be solved with respect to all accelerations, and there are $n - r$ functionally independent constraints

$$\varphi_\alpha(q, \dot{q}) = 0, \qquad 1 \leq \alpha \leq n - r$$

If these constraints cannot be integrated (reduced to ones depending on the coordinates only), the mechanics is nonholonomic.

Remark. If Lagrangian $L_0(q, \dot{q})$ is regular with externally imposed holonomic constraints $\varphi_\alpha(q) = 0$, the system is equivalent to the singular one with Lagrangian $L = L_0 + \lambda_\alpha \varphi_\alpha$ and extra generalized coordinates λ_α.

Contents

Dirac's Hamiltonian Formalism

Aimed at quantisation of gauge systems.

Passing to the Hamiltonian description via a Legendre transformation

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

the degeneracy of the Hessian H_{ij} manifests itself in the existence of $n - r$ relations between coordinates and momenta, the set Σ_1 of primary constraints

$$\Sigma_1 := \{\, \phi_\alpha^{(1)}(p, q) = 0 \mid 1 \leq \alpha \leq n - r \,\} .$$

The dynamics is constrained by the set Σ_1 and is governed by the total Hamiltonian

$$H_T := H_C + U_\alpha \phi_\alpha^{(1)} ,$$

where $H_C(p, q) := p_i q_i - L$ is the canonical Hamiltonian and U_α are Lagrange multipliers.

Consistency Conditions

Hamiltonian equations are given by

$$\dot{q}_i = \{H_T, q_i\}, \quad \dot{p}_i = \{H_T, p_i\}, \quad \phi_\alpha^{(1)}(p, q) = 0$$

with Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial p}{\partial q_i}$$

The primary constraints must satisfy the consistency conditions

$$\dot{\phi}_\alpha^{(1)} = \{H_T, \phi_\alpha^{(1)}\} \stackrel{\Sigma_1}{=} 0 \quad (1 \le \alpha \le n - r)$$

$\stackrel{\Sigma_1}{=}$ means the equality modulo the set of primary constraints.

Complete Set of Constraints

The consistency condition for $\phi_\alpha^{(1)}(p,q)$, unless it is satisfied identically, lead to one of the alternatives:

1. Contradiction $\Longleftrightarrow$ inconsistency.
2. New constraint. If it does not involve U_α, it is called secondary constraint and must be added to the constraint set.

The iteration of the consistency check ends up with the complete set of constraints

$$\Sigma := \{\ \phi_\alpha(p,q) = 0 \ \mid\ 1 \le \alpha \le k\ \}$$

which contains primary $\phi_\alpha^{(1)}(p,q)$, secondary $\phi_\alpha^{(2)}(p,q)$, ternary $\phi_\alpha^{(3)}(p,q)$, etc., constraints.

Remark. Secondary, etc., constraints are integrability conditions of the Hamiltonian system, and their incorporation is completion to involution (Hartley, Tucker, Seiler)

Constraints of First and Second Classes

The co-rank $s := k - rank(\mathbb{M})$ of the Poisson bracket matrix

$$\mathbb{M}_{\alpha\beta} :\stackrel{\Sigma}{=} \{\phi_\alpha, \phi_\beta\} ,$$

represent the number of first-class constraints $\psi_1 , \psi_2 , \dots , \psi_s$. Generally, they are linear combinations of constraints ϕ_α

$$\psi_\alpha(p, q) = \sum_\beta \mathrm{c}_{\alpha\beta}(p, q)\, \phi_\beta ,$$

whose Poisson brackets are zero modulo the constraints set

$$\{\psi_\alpha(p, q), \psi_\beta(p, q)\} \stackrel{\Sigma}{=} 0 \qquad 1 \le \alpha , \beta \le s .$$

The remaining functionally independent constraints form the subset of second-class constraints.

Gauge Transformations

First-class constraints play a very special role in the Hamiltonian description: they generate gauge symmetry.

By Dirac's conjecture, the generator G of gauge transformations is expressed as a linear combination of the first-class constraints

$$G = \sum_{\alpha=1}^{s} \varepsilon_\alpha \psi_\alpha(p, q)$$

where the coefficients ε_α are functions of t.

The generator G must be conserved modulo the primary constraints

$$\frac{dG}{dt} \stackrel{\Sigma_1}{=} 0$$

and its action on phase space coordinates (p, q), in the presence of the first-class constraints only, is given by

$$\delta q_i = \{G, q_i\}, \qquad \delta p_i = \{G, p_i\}\,.$$

Physical Observables

Physical requirement: observables are invariant (singlets) under the gauge symmetry transformations.

This requirement has direct impact on the Hamiltonian reduction, that is a formulation of a new Hamiltonian system with a reduced number of degrees of freedom but equivalent to the initial degenerate one.

The presence of s first-class constraints and $r := k - s$ second-class constraints guarantees the possibility of local reformulation of the initial $2n$ dimensional Hamiltonian system as a $2n - 2s - r$ dimensional reduced (unconstrained) Hamiltonian system.

Remark. The reduced Hamiltonian system admits the canonical quantisation by imposing the standard commutation relations on the phase space variables.

Algorithmisation Issues

1. Compute all primary constraints
2. Determine all integrability conditions (secondary constraints) and separate them into first and second classes.
3. Construct the gauge symmetries generator and the basis for singlet observables
4. Find an equivalent unconstrained Hamiltonian system on the reduced phase space

Assumption. Hereafter we consider dynamical systems whose Lagrangians are polynomials in coordinates and velocities with rational (possibly parametric) coefficients

$$L(q, \dot{q}) \in \mathbb{Q}[q, \dot{q}]$$

Under this assumption issues I-II and the first part of issue III admit the complete algorithmisation.

Contents

Primary Constraints and Canonical Hamiltonian: algorithm

1. Use relations $p_i := \partial L / \partial \dot{q}_i$ as generators of polynomial ideal in $\mathbb{Q}[p, q, \dot{q}]$

$$I_{p,q,\dot{q}} := \mathrm{Id}(\cup_{i=1}^{n} \{p_i - \partial L / \partial \dot{q}_i\}) \subset \mathbb{Q}[p, q, \dot{q}]$$

2. Construct Gröbner basis (Buchberger) or involutive basis (Gerdt,Blinkov) $GB(I_{p,q,\dot{q}})$ by using an appropriate term ordering which eliminates $\dot{q}$, and take the intersection

$$GB(I_{p,q}) = GB(I_{p,q,\dot{q}}) \cap \mathbb{Q}[p, q]$$

3. Extract a subset $\Phi_1 \subset GB(I_{p,q})$ of algebraically independent primary constraints satisfying

$$\forall \phi(p,q) \in \Phi_1 \; : \; \phi(p,q) \notin \mathrm{Id}(\Phi_1 \setminus \{\phi(p,q)\})$$

   that is verified by the normal form $NF(\phi, GB(\mathrm{Id}(\Phi_1 \setminus \{\phi\}))$.

4. Compute $H_c(p,q) = NF(p_i q_i - L, GB(I_{p,q,\dot{q}}))$.

Contents

Complete Set of Constraints: algorithm

1. Compute Gröbner (involutive) basis GB of the ideal $\mathrm{Id}(\Psi) \subset Q[p,q]$ generated by $\Psi := \Phi_1$ in with respect to some ordering. Fix this ordering in the sequel.
2. Construct the total Hamiltonian $H_T = H_c + U_\alpha \phi_\alpha^{(1)}$ with Lagrange multipliers U_α treated as symbolic constants (parameters).
3. For every element $\phi_\alpha \in \Psi$ compute $h := NF(\{H_T, \phi_\alpha\}, GB)$. If $h \neq 0$ and no multipliers U_β occur in h, then enlarge set Ψ with h, and compute the Gröbner (involutive) basis GB for the enlarged set.
4. If $GB = \{1\}$, stop because the system is inconsistent. Otherwise, repeat the previous step until the consistency condition is satisfied for every element in Ψ irrespective of multipliers U_α.
5. Extract algebraically independent set $\Phi = \{\phi_1, \ldots, \phi_k\}$ from GB. This gives the complete set of constraints.

Contents

Separation of Constraints: algorithm

1. Construct the $k \times k$ Poisson bracket matrix as

$$\mathbb{M}_{\alpha,\beta} := NF(\{\phi_\alpha, \phi_\beta\}, GB)$$

2. Compute rank r of M.
   If $r = k$, stop with $\Phi_1 = \emptyset$, $\Phi_2 = \Phi$.
   If $r = 0$, stop with $\Phi_1 = \Phi$ and $\Phi_2 = \emptyset$.
   Otherwise, go to the next step.
3. Find a basis $A = \{a_1, \ldots, a_{k-r}\}$ of the null space (kernel) of $\mathbb{M}$. For every $a \in A$ construct a first-class constraint as $a_\alpha \phi_\alpha$. Collect them in set Φ_1.
4. Construct $(k - r) \times k$ matrix $(a_j)_\alpha$ from components of vectors in A and find a basis $B = \{b_1, \ldots, b_r\}$ of the null space of the corresponding linear transformation (cokernel of $\mathbb{M}$) . For every $b \in B$ construct a second-class constraint as $b_\alpha \phi_\alpha$. Collect them in set Φ_2.

Contents

Generator of Gauge Transformations

To eliminate the second-class constraints from consideration it is convenient to use Dirac bracket defined as

$$\{f,g\}_D := \{f,g\} - \{f,\chi_\alpha\}\mathbb{C}^{-1}_{\alpha\beta}\{\chi_\beta,g\}\,,$$

where χ_α $(1 \leq \alpha \leq r)$ denotes the second-class constraints, and the invertible $r \times r$ matrix $\mathbb{C}_{\alpha\beta}$ is defined as

$$\mathbb{C}_{\alpha\beta} \stackrel{\Sigma}{:=} \{\chi_\alpha,\chi_\beta\}$$

The gauge symmetry generator G is determined from the condition

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, H_C\}_D \stackrel{\Sigma_1}{=} 0$$

Remark. Since $\{f,\chi_\alpha\}_D = 0$ holds for an arbitrary function f, the second-class constraints can be set to zero either before or after evaluating a Dirac bracket. In particular, G acts on the canonical variables (p,q) as

$$\delta q_i = \{G, q_i\}_D, \qquad \delta p_i = \{G, p_i\}_D\,.$$

Generator of Gauge Transformations: algorithm

1. Compose the generator as a linear combination of the first-class constraints

$$G = \sum_{\beta=1}^{k_1} \varepsilon_\beta^{(1)}(t)\phi_\beta^{(1)} + \sum_{\gamma=k_1+1}^{s} \varepsilon_\gamma^{(2)}(t)\phi_\gamma^{(2)}\,, \quad \phi_\beta^{(1)} \in \Phi_1^{(1)},\ \phi_\gamma^{(2)} \in \Phi_1 \setminus \Phi_1^{(1)}$$

2. Construct Gröbner (involutive) basis GB_1 of the ideal $\mathrm{Id}(\Phi_1^{(1)})$
3. Compute $h := NF(\{G, H_c\}_D, GB_1)$
4. Express h in terms of $\phi_\gamma^{(2)}$. This yields

$$\frac{dG}{dt} \overset{\Sigma_1}{=} 0 \ \Rightarrow\ \dot{\varepsilon}_\gamma^{(2)}\phi_\gamma^{(2)} + \varepsilon_\beta^{(1)}\rho_{\beta\gamma}\phi_\gamma^{(2)} + \varepsilon_\delta^{(2)}\rho_{\delta\gamma}\phi_\gamma^{(2)} \overset{\Sigma_1}{=} 0, \quad \rho_{\mu\nu} \in \mathbb{Q}[p, q]$$

It follows: $\dot{\varepsilon}_\gamma^{(2)} + \varepsilon_\beta^{(1)}\rho_{\beta\gamma} + \varepsilon_\delta^{(2)}\rho_{\delta\gamma} = 0\ \ (k_1 + 1 \le \gamma \le s) \Longrightarrow \varepsilon^{(1)}$ is expressed in terms of $\varepsilon^{(2)}$ and $\dot{\varepsilon}^{(2)}$

Contents

Yang-Mills Theory

The standard action of the $SU(n)$ Yang-Mills field theory in Minkowski space M_4, endowed with a metric η is

$$S := \frac{1}{g_0^2} \int_{M_4} \mathrm{tr}\, F \wedge *F\,, \quad F := dA + A \wedge A\,, \quad *F_{\mu\nu} := \frac{1}{2} \sqrt{\det(\eta)}\, \epsilon_{\mu\nu\alpha\beta}\, F^{\alpha\beta}$$

Here $A = A^a\, T^a\,,\ F = F^a\, T^a\ \ (a = 1, 2, \ldots, n^2 - 1)$. The light-cone coordinates $x^\mu = \left(x^+, x^-, x^\perp\right)$ are given by

$$x^\pm := \frac{1}{\sqrt{2}} \left(x^0 \pm x^3\right)\,, \quad x^\perp := x^k\,, \quad k = 1, 2\,.$$

non-zero components of η are $\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1\,.$

In the light-cone Yang-Mills mechanics the components in the connection one-form $A := A_+\, dx^+ + A_-\, dx^- + A_k\, dx^k$ depend only on the light-cone “time variable” x^+, i.e., $A_\pm = A_\pm(x^+)\,,\ A_k = A_k(x^+)\,.$

Lagrangian

Lagrangian of the light-cone Yang-Mills mechanics

$$L := \frac{1}{2g^2}\left(F^a_{+-}\,F^a_{+-} + 2\,F^a_{+k}\,F^a_{-k} - F^a_{12}\,F^a_{12}\right)\,,$$

Here g is the “renormalized” coupling constant, and the components of the field-strength tensor are given by

$$\begin{aligned}
F^a_{+-} &:= \frac{\partial A^a_-}{\partial x^+} + \mathrm{f}^{abc}\,A^b_+\,A^c_-\,,\\
F^a_{+k} &:= \frac{\partial A^a_k}{\partial x^+} + \mathrm{f}^{abc}\,A^b_+\,A^c_k\,,\\
F^a_{-k} &:= \mathrm{f}^{abc}\,A^b_-\,A^c_k\,,\\
F^a_{ij} &:= \mathrm{f}^{abc}\,A^b_i\,A^c_j\,, \quad i,j,k = 1,2
\end{aligned}$$

and f^{abc} are the structure constants of $SU(n)$.

Hamiltonian Formulation

The Legendre transformation

$$\pi_a^{+} := \frac{\partial L}{\partial \dot{A}_{+}^{a}} = 0\,,$$
$$\pi_a^{-} := \frac{\partial L}{\partial \dot{A}_{-}^{a}} = \frac{1}{g^2}\,\left(\dot{A}_{-}^{a} + \mathrm{f}^{abc}\,A_{+}^{b}\,A_{-}^{c}\right)\,,$$
$$\pi_a^{k} := \frac{\partial L}{\partial \dot{A}_{k}^{a}} = \frac{1}{g^2}\,\mathrm{f}^{abc}\,A_{-}^{b}\,A_{k}^{c}$$

gives the canonical Hamiltonian

$$H_C = \frac{g^2}{2}\,\pi_a^{-}\pi_a^{-} - \mathrm{f}^{abc}\,A_{+}^{b}\left(A_{-}^{c}\,\pi_a^{-} + A_{k}^{c}\,\pi_a^{k}\right) + \frac{1}{2g^2}F_{12}^{a}F_{12}^{a}\,.$$

The non-vanishing Poisson brackets between the canonical variables

$$\{A_{\pm}^{a}\,,\pi_b^{\pm}\} = \delta_b^a\,, \qquad \{A_k^a\,,\pi_b^l\} = \delta_k^l\delta_b^a$$

Primary and Some Secondary Constraints

$\det \|\frac{\partial^2 L}{\partial \dot{A} \partial \dot{A}}\| = 0$, and the primary constraints are

$$\begin{cases} \varphi_a^{(1)} := \pi_a^+ = 0 \\ \chi_k^a := g^2\, \pi_k^a + \mathrm{f}^{abc}\, A_-^b A_k^c = 0 \end{cases} \qquad \{\chi_i^a\,, \chi_j^b\} = \ 2\mathrm{f}^{abc} \eta_{ij} A_-^c$$

The total Hamiltonian $H_T := H_C + U_a \varphi_a^{(1)} + V_k^a \chi_k^a$ yields for $\varphi_a^{(1)}$

$$\dot{\varphi}_a^{(1)} = \{\pi_a^+\,, H_T\} = \mathrm{f}^{abc} \left(A_-^b \pi_c^- + A_k^b \pi_c^k \right) \stackrel{\Sigma_1}{=} 0$$

that generates $n^2 - 1$ secondary constraints

$$\varphi_a^{(2)} := \mathrm{f}_{abc} \left(A_-^b \pi_c^- + A_k^b \pi_c^k \right) = 0, \quad \{\varphi_a^{(2)}\,, \varphi_b^{(2)}\} = \mathrm{f}_{abc}\, \varphi_c^{(2)}$$

The same procedure for χ_k^a gives the consistency conditions

$$\dot{\chi}_k^a = \{\chi_k^a\,, H_C\} - 2\, g^2\, \mathrm{f}^{abc}\, V_k^b\, A_-^c \stackrel{\Sigma_1}{=} 0$$

The further analysis depends on n.

Contents

Constraints and Their Separation

For $SU(2)$: $\mathrm{f}^{abc} := \epsilon^{abc}$. The complete set of constraints contains 9 primary constraints $\varphi_a^{(1)}, \chi_k^a$ and 3 secondary ones $\varphi_a^{(2)}$. Separation of the primary constraints gives 2 additional first-class constraints

$$\psi_k := A_-^a \chi_k^a \,,$$

and 4 second-class constraints

$$\chi_{k\perp}^a := \chi_k^a - \frac{\left(A_-^b \chi_k^b\right)\, A_-^a}{(A_-^1)^2 + (A_-^2)^2 + (A_-^3)^2}$$

The new first-class constraints ψ_i are abelian, $\{\psi_i\,, \psi_j\} = 0$, and have also zero Poisson brackets with other constraints, while for the second-class constraints $\chi_{k\perp}^a$ non-zero Poisson brackets read

$$\{\chi_{i\perp}^a\,, \chi_{j\perp}^b\} = 2\,\epsilon^{abc}\, A_-^c\, \delta_{ij}\,,$$
$$\{\varphi_a^{(2)}\,, \chi_{k\perp}^b\} = \epsilon^{abc}\, \chi_{k\perp}^c\,.$$

Thus, there are 8 first-class constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$ and 4 second-class constraints $\chi_{k\perp}^a$.

Generator of Gauge Transformations

Generator G is sought as

$$G = \sum_{a=1}^{3} \varepsilon_a^{(1)} \varphi_a^{(1)} + \sum_{i=1}^{2} \eta_i \psi_i + \sum_{a=1}^{3} \varepsilon_a^{(2)} \varphi_a^{(2)}$$

with 8 light-cone time-dependent functions $\varepsilon_a^{(1)}(\tau)$, $\varepsilon_a^{(2)}(\tau)$ and $\eta_i(\tau)$.
From the condition $dG/dt \stackrel{\Sigma_1}{=} 0$ it follows

$$\left(\dot{\varepsilon}_a^{(2)} + \varepsilon_a^{(1)} - \epsilon_{abc} \varepsilon_b^{(2)} A_+^c - \eta_i A_i^a\right) \phi_a^{(2)} \stackrel{\Sigma_1}{=} 0$$

Expressing $\varepsilon_a^{(1)}$ in terms of the functions $\varepsilon_a^{(2)}$ yields the final form of the generator

$$G = \left(-\dot{\varepsilon}_a^{(2)} + \epsilon_{abc} \varepsilon_b^{(2)} A_+^c + \eta_i A_i^a\right) \phi_a^{(1)} + \eta_i \psi_i + \varepsilon_a^{(2)} \phi_a^{(2)}$$

Hamiltonian Reduction

Unconstrained phase space has dimension 4. The reduction to this space is done as follows (Gerdt, Khvedelidze, Mladenov).

Gauge degrees of freedom associated with $\varphi_a^{(1)}$ are trivially eliminated since A_+^b play role of Lagrange multipliers in H_T for $\varphi_a^{(2)}$ and dropped out after the projection to the constraint shell.

To eliminate the gauge degrees associated with $\varphi_a^{(2)}$ construct 3×3 matrix $A := \|A_1^a\,, A_2^a\,, A_-^a\|$ and use polar representation

$$A = OS$$

where S is a positive definite 3×3 symmetric matrix and O is the orthogonal matrix parameterized by 3 Euler angles. These angles are just the pure gauge degrees of freedom corresponding to the constraints $\varphi_a^{(2)}$.

Hamiltonian Reduction (cont.)

To eliminate the gauge degrees connected with the remaining two abelian constraints ψ_1, ψ_2 one can pass to a principal axes representation for the symmetric matrix S

$$S = R^T \operatorname{diag}(q_1, q_2, q_3)\, R$$

with the orthogonal matrix $R(\theta_1, \theta_2, \theta_3)$ given in terms of the Euler angles $(\theta_1, \theta_2, \theta_3)$. Now again it turns out that the two angles θ_1 and θ_2 are pure gauge degrees of freedom.

Solving for the remaining second class constraints $\chi^a_{i\,\perp}$ leads to an unconstrained model, so-called conformal mechanics with phase space variables (q_1, p_1) and (θ_3, p_{θ_3}) whose Hamiltonian

$$H = \frac{g^2}{2}\left(p_1^2 + \frac{p_{\theta_3}^2}{4}\,\frac{1}{q_1^2}\right)$$

is a projection of H_c to the constraints shell.

Conclusions

- Dirac's Hamiltonian formalism for degenerate mechanical systems with polynomial Lagrangians admit full algorithmisation of the following steps: computation and separation of the complete set of constraints and construction of the gauge symmetry generator.
- Gröbner or involutive bases form the fundamentals of the algorithmisation since these bases allow to work algorithmically modulo constraints.
- Algorithmisation of determination of the basis for unconstrained observables and of the Hamiltonian reduction to these observables still remain to be done.
- For the $SU(2)$ Yang-Mills light-cone mechanics the Hamiltonian reduction has been performed, and the reduced model is conformal mechanics.