The Homogeneous Gröbner Basis for the *SU*(3)-gauge Mechanics

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$SU(3)$ Yang-Mills Theory

The Lagrangian

$$L = -\frac{1}{4}\, F^a_{\mu\nu}\, F_a^{\mu\nu}\,,$$

the components of the field-strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \mathrm{f}^{abc}\, A^b_\mu A^c_\nu$$

The commutator of the Gell-Mann matrices

$$\{\lambda_a, \lambda_b\} = 2\,\mathrm{i}\,\mathrm{f}^{abc}\lambda_c$$

where f^{abc} are the structure constants of $SU(3)$.

The Light-Cone

The Cartesian coordinates: $x^\mu = \left(x^0, x^1, x^2, x^3\right)$
Minkowski metric tensor: $\eta = \text{diagonal}(+1, -1, -1, -1)$.

The light-cone: coordinates $x^\mu = \left(x^+, x^-, x^\perp\right)$

$$x^\pm := \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right)\,, \quad x^\perp := x^k\,, \quad k = 1, 2\,.$$

metric: $\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1\,.$

The light-cone Yang-Mills mechanics: the fields depend only on the light-cone "time variable" x^+

$$A^a_\pm = A^a_\pm(x^+)\,,\ A^a_k = A^a_k(x^+)\,.$$

Lagrangian of the light-cone Yang-Mills mechanics

$$L := \frac{1}{2}\left(F^a_{+-}\,F^a_{+-} + 2\,F^a_{+k}\,F^a_{-k} - F^a_{12}\,F^a_{12}\right)\,,$$

$$\begin{aligned}
F^a_{+-} &:= \dot{A}^a_{-} + \mathrm{f}^{abc}\,A^b_{+}\,A^c_{-}\,,\\
F^a_{+k} &:= \dot{A}^a_{k} + \mathrm{f}^{abc}\,A^b_{+}\,A^c_{k}\,,\\
F^a_{-k} &:= \mathrm{f}^{abc}\,A^b_{-}\,A^c_{k}\,,\\
F^a_{ij} &:= \mathrm{f}^{abc}\,A^b_{i}\,A^c_{j}\,, \qquad i,j,k = 1,2\,,
\end{aligned}$$

$$\dot{A}^a_{\mu} := \partial A^a_{\mu}/\partial x^{+}$$

The system is degenerated

$$\left|\frac{\partial^2 L}{\partial \dot{A}\partial \dot{A}}\right| = 0$$

Hamiltonian Formulation

The Legendre transformation

$$\pi_a^{+} := \frac{\partial L}{\partial \dot{A}_{+}^{a}} = 0\,,$$

$$\pi_a^{-} := \frac{\partial L}{\partial \dot{A}_{-}^{a}} = \dot{A}_{-}^{a} + \mathrm{f}^{abc} A_{+}^{b} A_{-}^{c}\,,$$

$$\pi_a^{k} := \frac{\partial L}{\partial \dot{A}_{k}^{a}} = \mathrm{f}^{abc} A_{-}^{b} A_{k}^{c}$$

gives the canonical Hamiltonian

$$H_C = \frac{1}{2}\pi_a^{-}\pi_a^{-} - \mathrm{f}^{abc} A_{+}^{a}\left(A_{-}^{b}\pi_c^{-} + A_k^{b}\pi_c^{k}\right) + \frac{1}{2}F_{12}^{a}F_{12}^{a}\,.$$

The canonical Poisson brackets

$$\{A_{\pm}^{a}\,,\pi_b^{\pm}\} = \delta_b^a\,, \qquad \{A_i^a\,,\pi_b^j\} = \delta_i^j\delta_b^a$$

Dirac procedure for a constrained theory

The primary constraints

$$\varphi_a^{(1)} := \pi_a^+ = 0\,, \qquad \chi_k^a := \pi_k^a + \mathrm{f}^{abc} A_-^b A_k^c = 0$$

have kinematical character and must be preserved during the evolution governed by the total Hamiltonian

$$H_T := H_C + u_a \varphi_a^{(1)} + v_k^a \chi_k^a\,,$$

so the trajectories remain on the constraint surface Σ all time,

$$\dot{\varphi}_\alpha^{(1)} = \{\varphi_\alpha^{(1)}, H_T\} \stackrel{\Sigma}{=} 0\,, \qquad \dot{\chi}_k^a = \{\chi_k^a, H_T\} \stackrel{\Sigma}{=} 0.$$

$\stackrel{\Sigma}{=}$: the right-hand side of vanishes modulo the constraints.

$$\dot{\varphi}_a^{(1)} = \{\pi_a^+, H_T\} = \mathrm{f}^{abc}\left(A_-^b \pi_c^- + A_k^b \pi_c^k\right) \stackrel{\Sigma}{=} 0$$

this generates secondary constraints

$$\varphi_a^{(2)} := \mathrm{f}^{abc}\left(A_-^b \pi_c^- + A_k^b \pi_c^k\right) = 0,$$

$$\dot{\varphi}_\alpha^{(2)} = \{\varphi_\alpha^{(2)}, H_T\} \stackrel{\Sigma}{=} 0.$$

The same procedure for χ_k^a gives the consistency conditions

$$\dot{\chi}_i^a = \{\chi_i^a, H_C\} + \{\chi_i^a, \chi_j^b\} v_j^b \overset{\Sigma}{=} 0$$

$$\{\chi_i^a, \chi_j^b\} = \mathrm{M}_{ab}\, \delta_{ij}\,, \qquad \mathrm{M}_{ab} := 2\, \mathrm{f}^{abc} A_-^c\,.$$

$$\det \mathrm{M} = 0,$$

The further analysis depends on n in $SU(n)$-Yang-Mills theory.

$$\operatorname{rank} \mathrm{M} = (n^2 - 1) - \operatorname{rank} \mathrm{su}(n),$$

$$\text{at least for} \quad n = 2, 3, 4.$$

null-vectors of M for $SU(n)$

$n = 2, 3, 4, \ldots$

$$e_a^{(1)} := A_-^a\,, \qquad \psi_i := e_a^{(1)} \chi_i^a = A_-^a \pi_a^i\,,$$

$n = 3, 4, \ldots$

$$e_a^{(2)} := \mathrm{d}^{abc} A_-^b A_-^c\,, \qquad \varsigma_i := e_a^{(2)} \chi_i^a = \mathrm{d}^{abc} A_-^a A_-^b \pi_c^i\,.$$

$n = 4, \ldots$

$$e_a^{(3)} := \mathrm{d}^{abc} \mathrm{d}^{cde} A_-^b A_-^d A_-^e\,, \qquad \xi_i := e_a^{(3)} \chi_i^a = \mathrm{d}^{abc} \mathrm{d}^{cde} A_-^a A_-^b A_-^d \pi_e^i\,.$$

$$\chi_i^a := \pi_i^a + \mathrm{f}^{abc} A_-^b A_i^c$$

$$\chi_i^a \quad \to \quad (\chi_\perp, \psi_i, \varsigma_i)$$

$\chi_\perp$ are projections of χ_i^a by vectors, which are orthogonal to null-vectors of matrix $||\{\chi_i^a, \chi_j^b\}||$. The consistency conditions for $\chi_\perp$ determine correspondig Lagrangian multipliers $v_\perp$ in the total Hamiltonian.

While for ψ_i, ς_i the Dirac procedure should be continued.

$$\begin{aligned}\{\psi_i, H_T\} &= \pi_a^- \pi_a^i + \mathrm{f}^{abc} A_-^a A_j^b F_{ji}^c \\ &= -A_i^a \varphi_a^{(2)} + \pi_a^- \chi_i^a + \mathrm{f}^{abc} A_i^a A_k^b \chi_k^c \stackrel{\Sigma}{=} 0\end{aligned}$$

$$\{\varsigma_i, H_T\} \stackrel{\Sigma}{=} \zeta_i$$

$$\zeta_i = \mathrm{d}^{abc} A_i^a F_{-k}^b F_{-k}^c$$

Now the total Hamiltonian is

$$H_T = H_C + \chi_\perp v_\perp + \psi_k v_k^\psi + \varsigma_k\, v_k^\varsigma + \overset{(1)}{\phi}_a\, u_a$$

where $v_k^\psi, v_k^\varsigma, u_a$ are still unknown Lagrangian multipliers.
The consistency condition

$$\{\zeta_i, H_T\} = \{\zeta_i, H_C\} + \{\zeta_i, \chi_\perp\}\, v_\perp + \{\zeta_i, \psi_k\}\, v_k^\psi + \{\zeta_i, \varsigma_k\}\, v_k^\varsigma \overset{\Sigma}{=} 0$$

where

$$\{\psi_i, \zeta_j\} = \delta_{ij}\, \mathrm{d}^{abc} A_-^a (F_{-k}^b \chi_k^c - \frac{1}{2} A_-^b \varphi_c^{(2)}) \overset{\Sigma}{=} 0$$

fixes v_k^ς because

$$\{\varsigma_i, \zeta_j\} = -\delta_{ij}\, \mathrm{d}^{abc} \mathrm{d}^{cpq} A_-^a A_-^b F_{-k}^p F_{-k}^q$$

is not zero on the constraint surface. So we have found all constraints.

The homogeneous Gröbner basis

With the *grading* Γ determined by the weights of the variables:

$$\Gamma(\pi_a^\mu) = 2\,, \qquad \Gamma(A_\mu^a) = 1\,, \qquad a = 1, 2, \ldots, 8\,, \qquad \mu = -\,, 1, 2\,,$$

we have the set of homogeneous polynomials ($k = 1, 2$)

$\Gamma -$ degree	Constraints
2	$\chi_k^a = \pi_a^k - \mathrm{f}^{abc} A_-^b A_k^c$
3	$\varphi_a^{(2)} = \mathrm{f}_{abc}\left(A_-^b \pi_c^- + A_k^b \pi_c^k\right)$
5	$\zeta_i = d_{abc} A_i^a F_{-k}^b F_{-k}^c$

The lexicographical order is

$$\pi_a^- \succ \pi_b^1 \succ \pi_c^2 \succ A_-^a \succ A_1^b \succ A_2^c \qquad a, b, c = 1, 2, \ldots, 8\,,$$

and for variables with the same spatial index μ we choose

$$\pi_a^\mu \succ \pi_b^\mu \succ A_\mu^a \succ A_\mu^b \qquad \text{if} \quad a < b\,.$$

To simplifiy calculations we exlude some numerical cefficients by redefinition of variables

$$A^8_- \to A^8_-/\sqrt{3} \qquad P_8^- \to \sqrt{3}P_8^-$$
$$A^8_i \to A^8_i/\sqrt{3} \qquad P_8^i \to \sqrt{3}P_8^i$$

and multiplying of constraints by approciate factors

$$\chi^a_k \to 2 \times \chi^a_k \qquad \chi^8_k \to \chi^8_k/\sqrt{3}$$
$$\overset{(2)}{\phi}_a \to 2 \times \overset{(2)}{\phi}_a \qquad \overset{(2)}{\phi}_8 \to \overset{(2)}{\phi}_8 /\sqrt{3}$$

$$\zeta_i \to 8 \times \zeta_i$$

With such a choice of grading the constraints χ_k^a and $\varphi^{(2)}$ are the lowest degree homogeneous Gröbner basis elements G_2 and G_3 of the order 2 and 3, respectively. Higher degree elements of the basis are constructed step by step by doing the following manipulations:

(i) formation of all $S-$polynomials (G_i, G_j) ;

(ii) elimination of some superfluous $S-$polynomials according to the Buchberger's criteria;

(iii) computation of the normal forms of S-polynomials modulo the lower order elements with respect to the grading chosen.

The results of computation of the Gröbner basis elements of different orders n are shown in the following table where we explicitly indicated only $S-$polynomials with non-vanishing normal form.

G_n	Polynomials #	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	8	$\varphi_a^{(2)}$
G_4	15	$(G_3,\ G_3)$
G_5	14	$\zeta_i,\ (\zeta_i, G_j) \qquad i = 1, 2 \quad j = 2, 3, 4$ $(G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)$
G_6	13	$(G_2, G_5), (G_3, G_5), (G_4, G_5), (G_5, G_5)$ $(G_3, G_4), (G_4, G_4)$

With the lexicographical order

$$A_1^b \succ A_2^c \succ A_-^a \succ \pi_b^1 \succ \pi_c^2 \succ \pi_a^- \qquad a, b, c = 1, 2, \ldots, 8\,,$$

G_n	Polynomials #	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	72	$(G_2,\ G_2)$
G_4	176	$(G_2,\ G_3),\ (G_3,\ G_3)$
G_5	376	$(G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)$

G_3 contains:

$$\psi_i = A_-^a \chi_i^a\,, \qquad i = 1, 2$$

$$A_1^a\chi_1^a, \quad A_2^a\chi_2^a, \quad A_1^a\chi_2^a + A_2^a\chi_1^a\,.$$

ζ_i have other, "more simpler" form ($F^a_{-k} = \mathrm{f}^{abc} A_-^b\, A_k^c$)

$$\zeta_i = \mathrm{d}^{abc} A_i^a F^b_{-k} F^c_{-k} \qquad \longrightarrow \qquad \zeta_i = \mathrm{d}^{abc} A_i^a P_b^k P_c^k$$

The actual calculations were performed using the the computer algebra system *Mathematica* (version 5.0) running on the machine 2xOpteron-242 (1.6 Ghz) with 6Gb of RAM and have take about a month.
For the case of the structure group $SU(2)$ we used the built-in-function `GroebnerBasis` with monomial order `DegreeReverseLexicographic`

$$\{\pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2, \pi_3^1, \pi_3^2, \pi_1^-, \pi_2^-, \pi_3^-, A_1^1, A_2^1, A_1^2, A_2^2, A_1^3, A_2^3, A_-^1, A_-^2, A_-^3\}\,.$$

In this case the construction of the *complete homogeneous Gröbner basis* of 64 elements takes about 60 seconds.

Conclusions

- Gröbner bases technique makes feasible computation of the complete set of constraints in the frame of Dirac's Hamiltonian formalism for degenerate mechanical systems. We have found a new set of constraints depending on the rank of the structure group.
- Due to the large number of variables and constraints the special homogeneous Groebner basis has been constructed in Mathematica codes.
- The main part of time was spend for calculations which explicitly break down the invariance of the gauge theory.