Towards an algorithmic construction of the ring of polynomial invariants for the quantum entanglement issue

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Entangled pure states of quantum systems

Hilbert space of the systems of n qubits (spin-$\frac{1}{2}$ quantum particle)

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n} \cong \mathbb{C}^{2^n}$$

Quantum state

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle$$

is called

$$\begin{array}{ll} \text{separable} & \text{if} \quad |\Psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle \\ \text{entangled} & \text{otherwise} \end{array}$$

Bell states: entangled pure states of 2 qubits

To be separable the state must allow the representation:

$$(\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$$|\Phi^+\rangle = |00\rangle + |11\rangle$$
$$|\Phi^-\rangle = |00\rangle - |11\rangle$$

For $|\Phi^+\rangle$ we must have $\alpha\gamma = \beta\delta = 1, \quad \alpha\delta = \beta\gamma = 0$, but this system has no solution.

Local and Nonlocal unitary operations

Nonlocal unitary operations act on $|\Psi\rangle$

$$U(2^n), \qquad U\,U^\dagger = \mathbb{I}_{2^n}$$

Local unitary operations act on each qubit separately

$$U(2)^n \cong \underbrace{U(2) \otimes \cdots \otimes U(2)}_{n}$$

The measure of entanglement $E(|\Psi\rangle)$ must be unchanged under the local unitary operation because the entanglement reflects nonlocal quantum correlations between subsystems.

Classification of orbits by polynomial invariants

The space of orbits under the action of the local transformation group

$$\mathcal{O} := \mathbb{C}^{2^n} / U(2)^n$$

is the main mathematical object in the entanglement problem. From coefficients of the state $|\Psi\rangle = \sum a_{i_1,\dots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle$ we can construct polynomial functions

$$f_j(a_{i_1,\dots,i_n}), \qquad j = 1, \dots, \dim \mathcal{O}$$

that are invariant on each orbit (are non-local parameters). Theorems from invariant theory guarantee that a finite set of such polynomials is enough to distinguish the generic orbits under local transformation. Any good measure of entanglement $E(|\Psi\rangle)$ should be a function of non-local parameters.

Mixed state, density operator

Density operator for the mixed state of a 4-level quantum system

$$\rho = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\,\vec{n}\cdot\vec{\lambda}\right) \in M_4(\mathbb{C}).$$

Coherence (Bloch) vector $\vec{n} = \{n_1, \ldots, n_{15}\} \in \mathbb{R}^{15}$, $0 \leq \vec{n}\cdot\vec{n} \leq 1$.

Lambda-matrices form the basis of the Lie algebra $\mathfrak{su}(4)$

$$\lambda_i\lambda_j = \delta_{ij}\mathbb{I}_4 + (d_{ijk} + i\,f_{ijk})\lambda_k.$$

Properties of ρ-matrix following from its probabilistic meaning:

1. $\rho = \rho^\dagger$ Hermitian,
2. $\mathrm{Tr}(\rho) = 1$,
3. $\rho \geq 0$ positivity.

2-Qubit system

$$\rho = \frac{1}{4}\left(\mathbb{I}_2 \otimes \mathbb{I}_2 + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{I}_2 + \mathbb{I}_2\otimes\vec{b}\cdot\vec{\sigma} + c_{ij}\,\sigma_i\otimes\sigma_j\right) \in M_4(\mathbb{C})$$

$\vec{a}, \vec{b}$ are Bloch vectors of qubits, Pauli matrices σ_i form the basis of the Lie algebra $\mathfrak{su}(2)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$c_{ij} - a_i b_j = 0 \quad \Rightarrow$ the state is separable : $\rho_{sep} = \sum_{k=1}^{K} q_k \rho_k^A \otimes \rho_k^B$

Varying correlation matrix c_{ij} we obtain (3×3)-dimensional family of 2-qubit mixed states, which are locally indistinguishable.

Local unitary operation

$$\rho' = k\rho k^{\dagger} \qquad k \in SU(2) \otimes SU(2)$$

does not change the entanglement.

Nonlocal unitary operation

$$\rho' = u\rho u^{\dagger} \qquad u \in SU(4)$$

does change the entanglement.

Orbit space for 2-Qubit system

$$u\lambda_i u^\dagger = R_{ij}\lambda_j, \qquad R_{ij} \in SO(15, \mathbb{R})$$

$$\mathcal{O} := \frac{\mathbb{R}^{15}}{SO(3) \otimes SO(3)}$$

$$\dim \mathcal{O} = 15 - 2 \times 3 = 9$$

Statement of the problem.
1. The ring of the local polynomial invariants.

Ronald C King and Trevor A Welsh, *Qubits and invariant theory*, Journal of Physics: Conference Series **30** (2006) pp. 1-8

The entanglement of a two-qubit system is a non-local property so that measures of entanglement should be independent of all local transformations of the two qubits separately. Since a mixed two-qubit system is described by its density matrix, its non-local entangling properties must be described by local invariants of the density matrix. A complete set of local invariants is provided by local invariants that are homogeneous polynomials, that is polynomials of fixed total degree in the elements of the density matrix.

R C King, T A Welsh, and P D Jarvis,
The mixed two-qubit system and the structure of its ring of local invariants,
Phys. A: Math. Theor. **40** (2007) pp. 10083-10108

The ring of local invariants is not freely generated by integrity basis

$$\mathcal{R}^G \cong: \mathcal{P}/\ker\phi$$

In order to identify the $\ker\phi$ of the map $\phi: \ \mathcal{P} \to \mathcal{R}^G$ it is necessary to find the syzygies of first kind. $\mathcal{R}^G$ has the structure of the Cohen-Macaulay ring (and is also Gorenstein)

$$\mathcal{R}^G = \bigoplus_{k=0}^{15} J_k \, \mathbb{C}[K_1, \dots, K_{10}].$$

While the methods employed here in the mixed two-qubit case may be used more generally in the mixed N-qubit case, the calculations do not appear to be tractable at present even for $N = 3$, let alone for $N > 3$.

Statement of the problem.
2. Hermicity of density operator $\rho = \rho^\dagger$.

Characteristic equation for a 4×4 matrix $\vec{n} \cdot \vec{\lambda} \in \mathfrak{su}(4)$:

$$\chi(x) = x^4 + \alpha x^2 + \beta x + \gamma = 0$$

where

$$\alpha = -\vec{n} \cdot \vec{n}, \quad \beta = -\frac{2}{3} d_{ijk} n_i n_j n_k,$$
$$\gamma = \det \vec{n} \cdot \vec{\lambda} = \frac{1}{4}((\vec{n} \cdot \vec{n})^2 - 2\, d_{ijk} d_{lmk} n_i n_j n_l n_m).$$

Hermicity of a matrix leads to reality of its eigenvalues, and hence to positivity of discriminant of $\chi(x)$:

$$D(\chi) = -\beta^2(4\alpha^3 + 27\beta^2) + 16\gamma(9\alpha\beta^2 + (\alpha^2 - 4\gamma)^2) \geq 0$$

which is 12 degree w.r.t. n_i.

Statement of the problem.
3. Positivity of density operator $\rho \geq 0$.

Characteristic equation for a $N \times N$ Hermitian matrix ρ

$$|\mathbb{I}_N x - \rho| = x^N - S_1 x^{N-1} + \ldots + (-1)^N S_N = 0$$

where

$$S_k = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \rho \begin{pmatrix} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{pmatrix}$$

is the sum of the principal minors of order k.

Theorem

$$\rho \geq 0 \quad \Leftrightarrow \quad S_k \geq 0 \text{ for all } k.$$

Positive semidefiniteness of ρ defines the region of permissible values of the Bloch vector components. This set of inequalities and the ring of the local unitary invariants together represent semialgebraic set.

Using Viète formulas we have (note $(\vec{n} * \vec{n})_k = \sqrt{\frac{3}{2}}\, d_{ijk} n_i n_j$)

$$
\begin{aligned}
S_1 &= \mathrm{Tr}(\rho) = x_1 + x_2 + x_3 + x_4 = 1 \\
S_2 &= x_1 x_2 + x_1 x_3 + \ldots = \tfrac{3}{8}(1 - \vec{n} \cdot \vec{n}) \\
S_3 &= x_1 x_2 x_3 + \ldots = \tfrac{1}{16}(1 - 3\,\vec{n} \cdot \vec{n} + 2\,(\vec{n} * \vec{n}) \cdot \vec{n}) \\
S_4 &= \det \rho = x_1 x_2 x_3 x_4 \\
&= \tfrac{1}{256}((1 - 3\,\vec{n} \cdot \vec{n})^2 + 8\,(\vec{n} * \vec{n}) \cdot \vec{n} - 12\,(\vec{n} * \vec{n}) \cdot (\vec{n} * \vec{n}))
\end{aligned}
$$

Using S_1 we find that the conditional extremum for other S_k attains at $x_1 = x_2 = x_3 = x_4$. We get the set of inequalities:

$$
\begin{aligned}
&0 \leq \vec{n} \cdot \vec{n} \leq 1, \\
&0 \leq 1 - 3\,\vec{n} \cdot \vec{n} + 2\,(\vec{n} * \vec{n}) \cdot \vec{n} \leq 1, \\
&0 \leq (1 - 3\,\vec{n} \cdot \vec{n})^2 + 8\,(\vec{n} * \vec{n}) \cdot \vec{n} - 12\,(\vec{n} * \vec{n}) \cdot (\vec{n} * \vec{n}) \leq 1.
\end{aligned}
$$

- 2-nd order

$$C^{(200)} = a_i a_i, \quad C^{(020)} = b_i b_i, \quad C^{(002)} = c_{ij} c_{ij},$$

- 3-rd order

$$C^{(111)} = a_i b_j c_{ij}, \quad C^{(003)} = \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} c_{i\alpha} c_{j\beta} c_{k\gamma},$$

- 4-th order

$$\begin{aligned} C^{(202)} &= a_i a_j c_{i\alpha} c_{j\alpha}, \\ C^{(022)} &= b_\alpha b_\beta c_{i\alpha} c_{i\beta}, \\ C^{(004)} &= c_{i\alpha} c_{i\beta} c_{j\alpha} c_{j\beta}, \\ C^{(112)} &= \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} a_i b_\alpha c_{j\beta} c_{k\gamma}. \end{aligned}$$

Nonlocal invariants expressed in terms of the local invariants:

$$3\vec{n}\cdot\vec{n} = C^{(200)} + C^{(020)} + C^{(002)},$$

$$(\vec{n}*\vec{n})\cdot\vec{n} = C^{(111)} - \frac{1}{3!}C^{(003)},$$

$$\begin{aligned}6\,(\vec{n}*\vec{n})\cdot(\vec{n}*\vec{n}) =& 2(C^{(200)}C^{(020)} + C^{(202)} + C^{(022)} - C^{(112)}) \\ &+ (C^{(002)})^2 - C^{(004)}.\end{aligned}$$

All functionally independent local invariants are involved in inequalities.

Hermitian product and duality Dirac bra-ket vector spaces

The Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$$

is the space of Ket-vectors $|\xi\rangle$

$$|\xi\rangle \equiv \xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{C}^4.$$

Bra-vectors $\langle\xi| \equiv \bar{\xi} \in \mathcal{H}^*$ are complex conjugate to ξ. Fixed vector $\bar{\xi}$ may be considered as a linear form on $\mathcal{H}$ using Hermitian product $\langle\xi|\psi\rangle = \sum_i \bar{\xi}_i \psi^i$. Then $\mathcal{H}^*$ is the space of linear functionals on $\mathcal{H}$, i.e. the dual space. The point $\bar{\xi} \in \mathcal{H}^*$ defines a linear subspace (hyperplane)

$$\langle\xi|\psi\rangle = \sum_i \bar{\xi}_i \psi^i = 0$$

of points $\psi^i \in \mathcal{H}$.

Projective duality

$$\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{C}^4 \mapsto \xi = (\xi_0 : \xi_1 : \xi_2 : \xi_3) \in \mathbb{C}P^3$$

ket V	bra V^*
$P(V)$	P^*
hyperplane $\langle \xi \vert \psi \rangle$	point $\bar{\xi}$
point ψ	hyperplane $\psi^\vee$

where the hyperplane $\psi^\vee$ is the set of all hyperplanes in $P(V)$ passing through ψ.

I.M Gelfand, M.M. Kapranov, A.V. Zelevinsky *Dicriminants, Resultants, and Multidimensional Determinants*
In general, let $X \subset P(V)$ be a closed irreducible algebraic subvariety. A hyperplane $H \subset P(V)$ is said to be tangent to X if there exists a smooth point $x \in X$ such that $x \in H$ and the tangent space to H at x contains the tangent space to X at x. Denote by $X^\vee \subset P^*$ the closure set of all hyperplanes tangent to X. The variety $X^\vee$ is called projectively dual to X.

Let $F(A, x) = 0$, $A \in V^*$ to be a linear form on V. Analytically, the hyperplane $F(A, x) = 0$ belongs to $X^\vee$ if and only if $F(A, x)$ vanishes at some point of X with all its first derivatives:

$$\begin{cases} F(A, x) = 0 \\ \dfrac{\partial}{\partial x_{i_j}^{(j)}} F(A, x) = 0 \end{cases}$$

Example: 2 qubit system $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$

$$Z = z_0|00\rangle + z_1|01\rangle + z_2|10\rangle + z_3|11\rangle \qquad Z \in \mathcal{H}$$

For separable states

$$\begin{aligned} Z =& (x_0^{(1)}|0\rangle + x_1^{(1)}|1\rangle)(x_0^{(2)}|0\rangle + x_1^{(2)}|1\rangle) \\ &= x_0^{(1)}x_0^{(2)}|00\rangle + x_0^{(1)}x_1^{(2)}|01\rangle + x_1^{(1)}x_0^{(2)}|10\rangle + x_1^{(1)}x_1^{(2)}|11\rangle \end{aligned}$$

the set of eqn's for $x_{i_j}^{(j)}$ must be satisfied

$$\begin{aligned} x_0^{(1)}x_0^{(2)} = z_0, &\quad x_0^{(1)}x_1^{(2)} = z_1, \\ x_1^{(1)}x_0^{(2)} = z_2, &\quad x_1^{(1)}x_1^{(2)} = z_3. \end{aligned}$$

Consistency condition:

$$q = z_0 z_1 - z_2 z_3 = 0$$

defines a nonsingular quadric $q = Z^T Q Z$ ($\det Q \neq 0$):

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad Z = (z_0, z_1, z_2, z_3)$$

$q = 0$ is Segre variety X, the image of the Segre embedding

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^3.$$

The pure states off the quadric $q = 0$ are the entangled states.

Linear form on Segre variety X

$$F(A, x) = 0$$

$$F(A, x) = \sum_{i,j} a_{ij} z_{ij}$$

$$= a_{00} z_{00} + a_{01} z_{01} + a_{10} z_{10} + a_{11} z_{11}$$

$$= a_{00} x_0^{(1)} x_0^{(2)} + a_{01} x_0^{(1)} x_1^{(2)} + a_{10} x_1^{(1)} x_0^{(2)} + a_{11} x_1^{(1)} x_1^{(2)}$$

$$F(A, x) = {x^{(1)}}^T \cdot A \cdot x^{(2)} = (x_0^{(1)} \; x_1^{(1)}) \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} x_0^{(2)} \\ x_1^{(2)} \end{pmatrix}$$

$$
\begin{cases}
\dfrac{\partial}{\partial x_0^{(1)}} F(A,x) = a_{00}x_0^{(2)} + a_{01}x_1^{(2)} = 0 \\
\dfrac{\partial}{\partial x_1^{(1)}} F(A,x) = a_{10}x_0^{(2)} + a_{11}x_1^{(2)} = 0 \\
\dfrac{\partial}{\partial x_0^{(2)}} F(A,x) = a_{00}x_0^{(1)} + a_{10}x_1^{(1)} = 0 \\
\dfrac{\partial}{\partial x_1^{(2)}} F(A,x) = a_{01}x_0^{(1)} + a_{11}x_1^{(1)} = 0
\end{cases}
$$

Consistency condition:

$$
\mathrm{Det}A = a_{00}a_{11} - a_{01}a_{10} = 0
$$

$$
\mathrm{Det}A = 0 \quad \Rightarrow \quad F(A,x) = 0 \quad \text{is degenerated}
$$

$$
\begin{tikzcd}
\mathbb{C}^{k_1+1} \otimes \cdots \otimes \mathbb{C}^{k_n+1} \arrow[r, "\text{projectivization}"] \arrow[d] & \mathbb{C}P^{(k_1+1)\ldots(k_n+1)-1} \\
\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n} \arrow[r, "\text{Segre embedding}"'] & X \subset \mathbb{C}P
\end{tikzcd}
$$

pure states

Hyperdeterminants via Gröbner bases

$$F(A,x) = \sum_{i_1,\dots,i_n=0}^{k_1,\dots,k_n} a_{i_1,\dots,i_n}\, x^{(1)}_{i_1} \cdots x^{(n)}_{i_n}$$

$$\begin{cases} \dfrac{\partial}{\partial x^{(j)}_{i_j}} F(A,x) = 0 \qquad \text{for all } j, i_j \\[2ex] \dfrac{\partial^2}{\partial x^{(j_1)}_{i_{j_1}} \partial x^{(j_2)}_{i_{j_2}}} F(A,x) = 0 \\[2ex] \dots\dots\dots\dots\dots \\[2ex] \dfrac{\partial^{n-1}}{\partial x^{(j_1)}_{i_{j_1}} \dots \partial x^{(j_{n-1})}_{i_{j_{n-1}}}} F(A,x) = 0. \end{cases}$$

Example: 3 qubits ($2 \times 2 \times 2$)

$$F(A_3, x) = \sum a_{ijk}\, x_i^{(1)} x_j^{(2)} x_k^{(3)} = 0, \quad i, j, k = 0, 1$$

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \hookrightarrow X \subset \mathbb{C}P^7, \qquad X^\vee : \quad \mathrm{Det}\, A_3 = 0$$

where $\mathrm{Det}\, A_3 = 0$ is the consistency condition for the system

$$\frac{\partial}{\partial x_{i_j}^{(j)}} F(A, x) = 0 \qquad \text{for all } j, i_j$$

Second derivatives

$$\frac{\partial^2}{\partial x_{i_{j_1}}^{(j_1)} \partial x_{i_{j_2}}^{(j_2)}} F(A, x) = 0$$

define singularities of $X^\vee$. For example the system with $j_1 = 2,\ j_2 = 3$ has the following consistency condition: the matrix

$$\begin{pmatrix} a_{000} & a_{010} & a_{100} & a_{110} \\ a_{001} & a_{011} & a_{101} & a_{111} \end{pmatrix}$$

never has the full rank.

Minors

$$
\begin{aligned}
M_0 &= a_{000}a_{011} - a_{001}a_{010} \\
M_1 &= a_{000}a_{111} - a_{001}a_{110} + a_{100}a_{011} - a_{101}a_{010} \\
M_2 &= a_{100}a_{111} - a_{101}a_{110}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Det}\, A_3 =& M_1^2 - 4M_0M_2 \\
=& a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 \\
& - 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\
& \quad + a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\
& \quad + a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\
& + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})
\end{aligned}
$$

Stratification of the orbit space by singularities of hypedeterminants

A. Miyake:

The hierarchy of singularities of the hypedeterminant generates stratification of the orbits of local transformation group $SU(2)^n$, i.e. "onion" structure of entangled states :

$$S_{k+1} \supset S_k \supset \cdots \supset S_1 \supset S_0 = \oslash,$$

$$X \to S_1 \qquad X^\vee \to S_k$$

$S_j - S_{j-1} (j = 1, \ldots, k+1)$ give $k+1$ classes of entangled states. A set S_j of states of the local rank $\leq j$ is a closed subvariety under SLOCC and S_{j-1} is the singular locus of S_j.

- J.-G. Luque and J.-Y. Thibon (2002): hyperdeterminant for 4-qubit system ($2 \times 2 \times 2 \times 2$) is represented as polynomial function of local invariants;
- multipartite systems and multidimensional matrices

$$\rho = \frac{1}{8}\Big(\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2$$
$$+\vec{n}_a \cdot \vec{\sigma} \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{n}_b \cdot \vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \vec{n}_c \cdot \vec{\sigma}$$
$$+\vec{n}_{ab} \cdot \vec{\sigma} \otimes \vec{\sigma} \otimes \mathbb{I}_2 + \vec{n}_{ac} \cdot \vec{\sigma} \otimes \mathbb{I}_2 \otimes \vec{\sigma} + \vec{n}_{bc} \cdot \mathbb{I}_2 \otimes \vec{\sigma} \otimes \vec{\sigma}$$
$$+\vec{n}_{abc} \cdot \vec{\sigma} \otimes \vec{\sigma} \otimes \vec{\sigma}\Big)$$

$$\vec{n}_{ab} \cdot \vec{\sigma} \otimes \vec{\sigma} \equiv (n_{ab})_{ij}\, \sigma_i \otimes \sigma_j,$$
$$\vec{n}_{abc} \cdot \vec{\sigma} \otimes \vec{\sigma} \otimes \vec{\sigma} \equiv (n_{abc})_{ijk}\, \sigma_i \otimes \sigma_j \otimes \sigma_k.$$

Conclusions

- Mathematically, entanglement in a quantum system may be described by the structure of the orbits in state space under the action of the group $SU(2)^n$ of local transformations.
- Completely algorithmic procedure of computation of the ring of polynomial invariants of a continuous group is needed. Multidimensional matrices and Cayley hyperdeterminants naturally arise in this problem.
- The representation space of mixed states is a semi-algebraic set. The application of methods of the classical theory of invariants requires an additional analysis and modification of the known computational algorithms.
- The Gröbner bases technique may be introduced into entanglement problem to resolve the algorithmical and computational difficulties of the construction of the ring of polynomial invariants.