Exercises
on
Differential Geometry of Lie Groups

Contents

Chapter 1

SU(2) group

1.1 The group element and Maurer-Cartan forms

We take the SU(2)-group element in II type coordinates

$$U = A_3(\alpha)A_2(\beta)A_3(\gamma) = \exp(\tau_3\alpha)\exp(\tau_2\beta)\exp(\tau_3\gamma), \tag{1.1.1}$$

where τ_i matrices are given in appendix A and Euler angles have the values in ranges:

$$0 \le \alpha < 2\pi, \quad 0 \le \beta \le \pi, \quad 0 \le \gamma < 2\pi. \tag{1.1.2}$$

The matrix $U^\dagger$

$$U^\dagger = \mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha}. \tag{1.1.3}$$

is the Hermitian conjugated matrix to U

$$U\,U^\dagger = I_2. \tag{1.1.4}$$

1-parameter groups $A_i(t)$ are described by matrices

$$A_i(t) = \exp(t\tau_i) = I_2\cos\frac{t}{2} + 2\tau_i\sin\frac{t}{2} \tag{1.1.5}$$

and have following explicit forms

$$\exp(\tau_1 t) = \begin{pmatrix} \cos\frac{t}{2} & -i\sin\frac{t}{2} \\ -i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} \tag{1.1.6}$$

$$\exp(\tau_2 t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} \tag{1.1.7}$$

$$\exp(\tau_3 t) = \begin{pmatrix} e^{-\frac{i}{2}t} & 0 \\ 0 & e^{\frac{i}{2}t} \end{pmatrix}. \tag{1.1.8}$$

According to this the group element may be written in the form

$$U = \begin{pmatrix} \cos\frac{\beta}{2}\,\mathbf{e}^{-\frac{i}{2}(\alpha+\gamma)} & -\sin\frac{\beta}{2}\,\mathbf{e}^{-\frac{i}{2}(\alpha-\gamma)} \\ \sin\frac{\beta}{2}\,\mathbf{e}^{\frac{i}{2}(\alpha-\gamma)} & \cos\frac{\beta}{2}\,\mathbf{e}^{\frac{i}{2}(\alpha+\gamma)} \end{pmatrix}. \tag{1.1.9}$$

Left-invariant $\overset{\mathsf{L}}{\omega}$ and right-invariant $\overset{\mathsf{R}}{\omega}$ Maurer-Cartan forms are

$$\overset{\mathsf{L}}{\omega} := U^{-1}dU \qquad \overset{\mathsf{R}}{\omega} := dUU^{-1}. \tag{1.1.10}$$

Lie algebra valued Maurer-Cartan forms $\overset{\mathsf{L}}{\omega}$ and $\overset{\mathsf{R}}{\omega}$ may be decomposed using the basis $\{\tau_i\}$ of Lie algebra $\mathfrak{su}(2)$

$$\overset{\mathsf{L}}{\omega} = \tau_i \overset{\mathsf{L}}{\omega}{}^i(\alpha^\nu) \qquad \overset{\mathsf{R}}{\omega} = \tau_i \overset{\mathsf{R}}{\omega}{}^i(\alpha^\nu) \tag{1.1.11}$$

where we introduce $\mathbb{R}$ valued Maurer-Cartan forms $\overset{\mathsf{L}}{\omega}{}^i$ and $\overset{\mathsf{R}}{\omega}{}^i$

$$\overset{\mathsf{L}}{\omega}{}^i(\alpha^\nu) := \overset{\mathsf{L}}{\omega}{}^i_\mu(\alpha^\nu) d\alpha^\mu, \qquad \overset{\mathsf{R}}{\omega}{}^i(\alpha^\nu) := \overset{\mathsf{R}}{\omega}{}^i_\mu(\alpha^\nu) d\alpha^\mu \tag{1.1.12}$$

with notations[1]

$$\alpha^1 = \alpha, \qquad \alpha^2 = \beta, \qquad \alpha^3 = \gamma. \tag{1.1.13}$$

Let us deduce algebraical formulas for calculations of components $\overset{\mathsf{L}}{\omega}{}^i_\mu$ and $\overset{\mathsf{R}}{\omega}{}^i_\mu$. For $\mathbb{R}$-valued Maurer-Cartan forms (1.1.12) we have

$$\overset{\mathsf{L}}{\omega}{}^i = -2\,\mathrm{Tr}(\tau_i U^{-1} dU), \qquad \overset{\mathsf{R}}{\omega}{}^i = -2\,\mathrm{Tr}(\tau_i dU U^{-1}). \tag{1.1.14}$$

The differential of the U matrix can be expressed via differentials of the Euler parameters

$$dU = \partial_\mu U d\alpha^\mu \qquad \partial_\mu U := \frac{\partial U}{\partial \alpha^\mu}. \tag{1.1.15}$$

We introduce the following quantities

$$\overset{\mathsf{L}}{\Omega}_\mu := U^{-1}\partial_\mu U, \qquad \overset{\mathsf{R}}{\Omega}_\mu := (\partial_\mu U) U^{-1} \tag{1.1.16}$$

and derive formulas for components $\overset{\mathsf{L}}{\omega}{}^i_\mu$ and $\overset{\mathsf{R}}{\omega}{}^i_\mu$

$$\boxed{\overset{\mathsf{L}}{\omega}{}^i_\mu = -2\,\mathrm{Tr}(\tau_i \overset{\mathsf{L}}{\Omega}_\mu), \qquad \overset{\mathsf{R}}{\omega}{}^i_\mu = -2\,\mathrm{Tr}(\tau_i \overset{\mathsf{R}}{\Omega}_\mu)} \tag{1.1.17}$$

According to that Ω-matrices may be decomposed with respect to the basis $\{\tau_i\}$

$$\overset{\mathsf{L}}{\Omega}_\mu = \tau_i \overset{\mathsf{L}}{\omega}{}^i_\mu, \qquad \overset{\mathsf{R}}{\Omega}_\mu = \tau_i \overset{\mathsf{R}}{\omega}{}^i_\mu. \tag{1.1.18}$$

Maurer-Cartan forms in the natural basis $\{d\alpha^\mu\}$ have Ω-matrices as Lie algebra valued components

$$\overset{\mathsf{L}}{\omega} = \overset{\mathsf{L}}{\Omega}_\mu d\alpha^\mu, \qquad \overset{\mathsf{R}}{\omega} = \overset{\mathsf{R}}{\Omega}_\mu d\alpha^\mu, \tag{1.1.19}$$

Some properties of Ω-matrices are listed in appendix B.

Explicit expressions for $\mathbb{R}$ valued 1-forms (1.1.12) are

$$\begin{aligned} \overset{\mathsf{L}}{\omega}{}^1 &= -d\alpha\, \cos\gamma\, \sin\beta + d\beta\, \sin\gamma & \overset{\mathsf{R}}{\omega}{}^1 &= -d\beta\, \sin\alpha + d\gamma\, \cos\alpha\, \sin\beta \\ \overset{\mathsf{L}}{\omega}{}^2 &= d\alpha\, \sin\gamma\, \sin\beta + d\beta\, \cos\gamma & \overset{\mathsf{R}}{\omega}{}^2 &= d\beta\, \cos\alpha + d\gamma\, \sin\alpha\, \sin\beta \\ \overset{\mathsf{L}}{\omega}{}^3 &= d\alpha \cos\beta + d\gamma & \overset{\mathsf{L}}{\omega}{}^3 &= d\alpha + d\gamma \cos\beta \end{aligned} \tag{1.1.20}$$

[1]We will omit the dependence on Euler angles for brevity.

1.2 Invariant vectors and the adjoint matrix

Vectors $\{\xi_i\}$ are dual to 1-forms $\{\omega^i\}$

$$\langle \overset{\mathsf{L}}{\omega}{}^i, \overset{\mathsf{L}}{\xi}{}_j \rangle = \delta^i_j, \qquad \langle \overset{\mathsf{R}}{\omega}{}^i, \overset{\mathsf{R}}{\xi}{}_j \rangle = \delta^i_j, \tag{1.2.1}$$

where

$$\overset{\mathsf{L}}{\xi}{}_i := \overset{\mathsf{L}}{\xi}{}^\mu_i \, \partial_\mu, \qquad \overset{\mathsf{R}}{\xi}{}_i := \overset{\mathsf{R}}{\xi}{}^\mu_i \, \partial_\mu \tag{1.2.2}$$

and bilinear form $\langle *, * \rangle$ is defined by

$$\langle d\alpha^\mu, \partial_\nu \rangle = \partial_\nu \alpha^\mu = \delta^\mu_\nu. \tag{1.2.3}$$

According to duality we must invert matrices of 1-forms to get components of vectors

$$\boxed{\overset{\mathsf{L}}{\omega}{}^i_\mu \, \overset{\mathsf{L}}{\xi}{}^\mu_j = \delta^i_j \qquad \overset{\mathsf{R}}{\omega}{}^i_\mu \, \overset{\mathsf{R}}{\xi}{}^\mu_j = \delta^i_j} \tag{1.2.4}$$

The found solutions satisfy the definition of invariant vector fields

$$U\tau_i = \overset{\mathsf{L}}{\xi}{}_i U, \qquad \tau_i U = \overset{\mathsf{R}}{\xi}{}_i U. \tag{1.2.5}$$

Components of vectors allows to decompose τ_i in bases of matrices Ω (compare with (1.1.18))

$$\tau_i = \overset{\mathsf{L}}{\xi}{}^\mu_i \, \overset{\mathsf{L}}{\Omega}{}_\mu, \qquad \tau_i = \overset{\mathsf{R}}{\xi}{}^\mu_i \, \overset{\mathsf{R}}{\Omega}{}_\mu. \tag{1.2.6}$$

Explicit expressions for invariant vectors are

$$\begin{aligned}
\overset{\mathsf{L}}{\xi}{}_1 &= \frac{\cos\gamma}{\sin\beta}(\cos\beta\partial_3 - \partial_1) + \sin\gamma\,\partial_2 & \qquad \overset{\mathsf{R}}{\xi}{}_1 &= \frac{\cos\alpha}{\sin\beta}(\partial_3 - \cos\beta\,\partial_1) - \sin\alpha\,\partial_2 \\
\overset{\mathsf{L}}{\xi}{}_2 &= -\frac{\sin\gamma}{\sin\beta}(\cos\beta\,\partial_3 - \partial_1) + \cos\gamma\,\partial_2 & \qquad \overset{\mathsf{R}}{\xi}{}_2 &= \frac{\sin\alpha}{\sin\beta}(\partial_3 - \cos\beta\,\partial_1) + \cos\alpha\,\partial_2 \\
\overset{\mathsf{L}}{\xi}{}_3 &= \partial_3 & \qquad \overset{\mathsf{R}}{\xi}{}_3 &= \partial_1
\end{aligned} \tag{1.2.7}$$

From the definition of left/right invariant Maurer Cartan forms it follows that

$$\overset{\mathsf{L}}{\omega} = U^{-1} \, \overset{\mathsf{R}}{\omega} \, U \tag{1.2.8}$$

Let us define the adjoint matrix (R-matrix) by

$$U\tau_i U^\dagger = R_{ij}\tau_j. \tag{1.2.9}$$

R matrix belongs to $SO(3)$ group and its elements are

$$\boxed{R_{ij} = -2\,\mathrm{Tr}(U\tau_i U^\dagger \tau_j)} \tag{1.2.10}$$

From (1.1.10) and (1.2.5) it follows that

$$\overset{\mathsf{L}}{\omega}{}^i = R_{ij} \, \overset{\mathsf{R}}{\omega}{}^j, \qquad \overset{\mathsf{L}}{\xi}{}_i R_{ij} = \overset{\mathsf{R}}{\xi}{}_j. \tag{1.2.11}$$

$$\overset{\mathsf{L}}{\omega}{}^i_\mu = R_{ij} \, \overset{\mathsf{R}}{\omega}{}^j_\mu, \qquad \overset{\mathsf{L}}{\xi}{}^\mu_i R_{ij} = \overset{\mathsf{R}}{\xi}{}^\mu_j. \tag{1.2.12}$$

$$R_{ij} = \overset{\mathsf{L}}{\omega}{}^i_\mu \, \overset{\mathsf{R}}{\xi}{}^\mu_j = \overset{\mathsf{R}}{\omega}{}^i_\mu \, \overset{\mathsf{L}}{\xi}{}^\mu_j. \tag{1.2.13}$$

Matrix R_{ij} (i-rows, j-columns) looks like that

$$R_{ij} = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\beta\cos\gamma\sin\alpha + \cos\alpha\sin\gamma & -\cos\gamma\sin\beta \\ -\cos\gamma\sin\alpha - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} \tag{1.2.14}$$

1.3 Lie derivatives, Lie differential equations

If Lie derivative of a tensor with respect to a given vector is zero then this tensor is invariant with respect to the transformation of the manifold generated by this vector. So Lie derivatives ensure us that constructed forms and vectors are really left/right invariant. The Lie derivative of a tensor $T^{ab\ldots d}_{ef\ldots g}$ with respect to a vector field X has components

$$
\begin{aligned}
(\mathcal{L}_X T)^{ab\ldots d}_{ef\ldots g} = \left(\frac{\partial T^{ab\ldots d}_{ef\ldots g}}{\partial x^i}\right) X^i - \\
- T^{ib\ldots d}_{ef\ldots g}\frac{\partial X^a}{\partial x^i} - \ldots - T^{ab\ldots i}_{ef\ldots g}\frac{\partial X^d}{\partial x^i} + \\
+ T^{ab\ldots d}_{if\ldots g}\frac{\partial X^i}{\partial x^e} + \ldots + T^{ab\ldots d}_{ef\ldots i}\frac{\partial X^i}{\partial x^g}.
\end{aligned}
\tag{1.3.1}
$$

The Lie derivative of the vector field with respect to a given vector is the commutator of these vector fields

$$
\mathcal{L}_{\overset{L}{\xi}_i} \overset{L}{\xi}_j = [\overset{L}{\xi}_i, \overset{L}{\xi}_j] = \left(\overset{L}{\xi}{}^\mu_i \partial_\mu \overset{L}{\xi}{}^\nu_j - \overset{L}{\xi}{}^\mu_j \partial_\mu \overset{L}{\xi}{}^\nu_i\right)\partial_\nu
\tag{1.3.2}
$$

and may be derived from commutator of τ-matrices (see appendix A) using relation (1.2.5)

$$
[\overset{L}{\xi}_i, \overset{L}{\xi}_j] = \epsilon_{ijk}\, \overset{L}{\xi}_k, \qquad [\overset{R}{\xi}_i, \overset{R}{\xi}_j] = -\epsilon_{ijk}\, \overset{R}{\xi}_k.
\tag{1.3.3}
$$

In result, components of vectors must satisfy to the differential Lie equations:

$$
\overset{L}{\xi}{}^\mu_i (\partial_\mu \overset{L}{\xi}{}^\nu_j) - \overset{L}{\xi}{}^\mu_j (\partial_\mu \overset{L}{\xi}{}^\nu_i) - \epsilon_{ijk}\, \overset{L}{\xi}{}^\nu_k = 0,
\tag{1.3.4}
$$

$$
\overset{R}{\xi}{}^\mu_i (\partial_\mu \overset{R}{\xi}{}^\nu_j) - \overset{R}{\xi}{}^\mu_j (\partial_\mu \overset{R}{\xi}{}^\nu_i) + \epsilon_{ijk}\, \overset{R}{\xi}{}^\nu_k = 0.
\tag{1.3.5}
$$

Generators left/right translations are right/left invariant vectors. This means that the corresponding Lie derivatives (commutators) of left and right invariant vectors must be zero

$$
\overset{R}{\xi}{}^\mu_i (\partial_\mu \overset{L}{\xi}{}^\nu_j) - \overset{L}{\xi}{}^\mu_j (\partial_\mu \overset{R}{\xi}{}^\nu_i) = 0.
\tag{1.3.6}
$$

The left/right invariance of Maurer-Cartan forms demands

$$
(\mathcal{L}_{\overset{R}{\xi}_k} \overset{L}{\omega}{}^i)_\mu = \overset{R}{\xi}{}^\nu_k \partial_\nu \overset{L}{\omega}{}^i_\mu + \overset{L}{\omega}{}^i_\nu \partial_\mu \overset{R}{\xi}{}^\nu_k = 0,
\tag{1.3.7}
$$

$$
(\mathcal{L}_{\overset{L}{\xi}_k} \overset{R}{\omega}{}^i)_\mu = \overset{L}{\xi}{}^\nu_k \partial_\nu \overset{R}{\omega}{}^i_\mu + \overset{R}{\omega}{}^i_\nu \partial_\mu \overset{L}{\xi}{}^\nu_k = 0.
\tag{1.3.8}
$$

Let us take the Lie derivative of the duality relation (1.2.1)

$$
\mathcal{L}_{\overset{L}{\xi}_k} \langle \overset{L}{\omega}{}^i, \overset{L}{\xi}_j \rangle = \langle (\mathcal{L}_{\overset{L}{\xi}_k} \overset{L}{\omega}{}^i), \overset{L}{\xi}_j \rangle + \langle \overset{L}{\omega}{}^i, (\mathcal{L}_{\overset{L}{\xi}_k} \overset{L}{\xi}_j) \rangle = 0
\tag{1.3.9}
$$

Using

$$
\mathcal{L}_{\overset{L}{\xi}_k} \overset{L}{\omega}{}^i = \mathcal{L}_{\overset{L}{\xi}_k} (\overset{L}{\omega}{}^i_\mu dx^\mu) = (\mathcal{L}_{\overset{L}{\xi}_k} \overset{L}{\omega}{}^i_\mu) dx^\mu = (\overset{L}{\xi}{}^\nu_k \partial_\nu \overset{L}{\omega}{}^i_\mu + \overset{L}{\omega}{}^i_\nu \partial_\mu \overset{L}{\xi}{}^\nu_k) dx^\mu
\tag{1.3.10}
$$

and

$$
\mathcal{L}_{\overset{L}{\xi}_k} \overset{L}{\xi}_j = \epsilon_{kjl}\, \overset{L}{\xi}_l
\tag{1.3.11}
$$

we get

$$
\overset{L}{\xi}{}^\mu_i (\overset{L}{\xi}{}^\nu_j \partial_\nu \overset{L}{\omega}{}^k_\mu + \overset{L}{\omega}{}^k_\nu \partial_\mu \overset{L}{\xi}{}^\nu_j) - \epsilon_{ijk} = 0
\tag{1.3.12}
$$

and analogously

$$
\overset{R}{\xi}{}^\mu_i (\overset{R}{\xi}{}^\nu_j \partial_\nu \overset{R}{\omega}{}^k_\mu + \overset{R}{\omega}{}^k_\nu \partial_\mu \overset{R}{\xi}{}^\nu_j) + \epsilon_{ijk} = 0
\tag{1.3.13}
$$

1.4 Riemannian structures on group manifold

The interval

$$ds^2 = g_{\mu\nu} d\alpha^\mu d\alpha^\nu \tag{1.4.1}$$

on group manifold may be defined as

$$ds^2 := -\frac{1}{2}\mathcal{K}\left(\overset{\mathsf{L}}{\omega}, \overset{\mathsf{L}}{\omega}\right) \tag{1.4.2}$$

where $\mathcal{K}$ is Killing form (see appendix A). Using $\mathcal{K}(\tau_i, \tau_j) = -2\delta_{ij}$ we derive

$$ds^2 = -\frac{1}{2}\mathcal{K}(\tau_i, \tau_j)\, \overset{\mathsf{L}}{\omega}{}^i\, \overset{\mathsf{L}}{\omega}{}^j = \delta_{ij}\, \overset{\mathsf{L}}{\omega}{}^i\, \overset{\mathsf{L}}{\omega}{}^j. \tag{1.4.3}$$

$$ds^2 = -2\mathrm{Tr}\left((U^{-1}dU)(U^{-1}dU)\right) = -2\mathrm{Tr}\left((dUU^{-1})(dUU^{-1})\right) \tag{1.4.4}$$

$$ds^2 = -2\mathrm{Tr}\left(\overset{\mathsf{L}}{\Omega}_\mu\, \overset{\mathsf{L}}{\Omega}_\nu\right) d\alpha^\mu d\alpha^\nu = -2\mathrm{Tr}\left(\overset{\mathsf{R}}{\Omega}_\mu\, \overset{\mathsf{R}}{\Omega}_\nu\right) d\alpha^\mu d\alpha^\nu \tag{1.4.5}$$

$$\boxed{g_{\mu\nu} = \sum_i \overset{\mathsf{L}}{\omega}{}^i_\mu\, \overset{\mathsf{L}}{\omega}{}^i_\nu = \sum_i \overset{\mathsf{R}}{\omega}{}^i_\mu\, \overset{\mathsf{R}}{\omega}{}^i_\nu} \tag{1.4.6}$$

$$\boxed{g^{\mu\nu} = \sum_i \overset{\mathsf{L}}{\xi}{}^\mu_i\, \overset{\mathsf{L}}{\xi}{}^\nu_i = \sum_i \overset{\mathsf{R}}{\xi}{}^\mu_i\, \overset{\mathsf{R}}{\xi}{}^\nu_i} \tag{1.4.7}$$

$$\overset{\mathsf{L}}{\xi}{}^\mu_i = g^{\mu\nu}\,\delta_{ij}\, \overset{\mathsf{L}}{\omega}{}^j_\nu, \qquad \overset{\mathsf{R}}{\xi}{}^\mu_i = g^{\mu\nu}\,\delta_{ij}\, \overset{\mathsf{R}}{\omega}{}^j_\nu, \tag{1.4.8}$$

$$\overset{\mathsf{L}}{\omega}{}^i_\mu = g_{\mu\nu}\,\delta_{ij}\, \overset{\mathsf{L}}{\xi}{}^\nu_j, \qquad \overset{\mathsf{R}}{\omega}{}^i_\mu = g_{\mu\nu}\,\delta_{ij}\, \overset{\mathsf{R}}{\xi}{}^\nu_j, \tag{1.4.9}$$

Explicit form of the interval is

$$ds^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta\; d\alpha d\gamma \tag{1.4.10}$$

Its inverse is

$$g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{\sin^2\beta}\left(\partial_1^2 + \sin^2\beta\;\partial_2^2 + \partial_3^2 - 2\cos\beta\;\partial_1\partial_3\right) \tag{1.4.11}$$

Determinant of metric

$$\det g_{\mu\nu} = (\det g^{\mu\nu})^{-1} = (\det \overset{\mathsf{L}}{\omega}{}^i_\mu)^2 = (\det \overset{\mathsf{R}}{\omega}{}^i_\mu)^2 = \sin^2\beta \tag{1.4.12}$$

The invariant volume is

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sqrt{\det g_{\mu\nu}}\; d\alpha d\beta d\gamma = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin\beta\; d\alpha d\beta d\gamma = 8\pi^2 \tag{1.4.13}$$

Now we can directly calculate:

- Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}), \tag{1.4.14}$$

- Riemann tensor

$$R^{\alpha}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\beta} \tag{1.4.15}$$

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu}, \tag{1.4.16}$$

- Ricci tensor

$$\mathcal{R}_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta}, \tag{1.4.17}$$

- curvature scalar

$$\mathcal{R} = \mathcal{R}_{\mu\nu} g^{\mu\nu} = \frac{3}{2}. \tag{1.4.18}$$

We have verified that $SU(2)$ manifold is Einstein space:

$$\mathcal{R}_{\mu\nu} = \frac{\mathcal{R}}{n} g_{\mu\nu} \tag{1.4.19}$$

where $n = g^{\mu\nu} g_{\mu\nu} = 3$ for $SU(2)$ (this corresponds to $n = N^2 - 1$ for $SU(N)$),

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} g_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cos\beta \\ 0 & 1 & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}. \tag{1.4.20}$$

Calculation show that $SU(2)$ manifold is a space of constant curvature

$$\mathcal{R}_{\alpha\beta\gamma\delta} = K \begin{vmatrix} g_{\alpha\gamma} & g_{\alpha\delta} \\ g_{\beta\gamma} & g_{\beta\delta} \end{vmatrix} = K(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \tag{1.4.21}$$

where

$$K = \frac{\mathcal{R}}{n(n-1)}. \tag{1.4.22}$$

A space of constant curvature is conformally flat space, so its Weyl tensor

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{\mathcal{R}}{(n-1)(n-2)}(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) - \frac{1}{n-2}(g_{\alpha\mu}\mathcal{R}_{\beta\nu} - g_{\alpha\nu}\mathcal{R}_{\beta\mu} - g_{\beta\mu}\mathcal{R}_{\alpha\nu} + g_{\beta\nu}\mathcal{R}_{\alpha\mu}) \tag{1.4.23}$$

vanishes identically.

The covariant derivative of a tensor $T^{ab\ldots d}_{ef\ldots g}$ is

$$\nabla_h T^{ab\ldots d}_{ef\ldots g} = \frac{\partial T^{ab\ldots d}_{ef\ldots g}}{\partial x^h} + \Gamma^{a}_{hj} T^{jb\ldots d}_{ef\ldots g} \tag{1.4.24}$$

$$+\Gamma^{b}_{hj} T^{aj\ldots d}_{ef\ldots g} + \ldots + \Gamma^{d}_{hj} T^{ab\ldots j}_{ef\ldots g} - \tag{1.4.25}$$

$$-\Gamma^{j}_{he} T^{ab\ldots d}_{jf\ldots g} - \ldots - \Gamma^{j}_{hg} T^{ab\ldots d}_{ef\ldots j}. \tag{1.4.26}$$

As the covariant derivative of the metric is zero

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{1.4.27}$$

we get that $SU(2)$ manifold is a symmetric space, *i.e.*

$$\nabla_\lambda \mathcal{R}_{\mu\nu\rho\sigma} = 0. \tag{1.4.28}$$

Integral curves of the left/right invariant vector fields are simultaneously geodesic ones

$$\overset{\mathsf{L}}{\xi}{}^{\mu}_{i} \nabla_\mu \overset{\mathsf{L}}{\xi}_{i} = 0, \qquad \overset{\mathsf{R}}{\xi}{}^{\mu}_{i} \nabla_\mu \overset{\mathsf{R}}{\xi}_{i} = 0 \tag{1.4.29}$$

where the index i is fixed.

The Killing equation

$$\mathcal{L}_X g_{\mu\nu} \equiv X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu X^\lambda + g_{\lambda\mu}\partial_\nu X^\lambda = 0 \qquad (1.4.30)$$

or

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \qquad (1.4.31)$$

expresses the invariance of the metric with respect to the coordinate transformation generated by X. The constructed metric $g_{\mu\nu}$ is bi-invariant, so its Lie derivatives with respect to left/right invariant vectors are zero

$$\mathcal{L}_{\overset{\mathsf{L}}{\xi}_i} g_{\mu\nu} \equiv \overset{\mathsf{L}}{\xi}{}_i^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \overset{\mathsf{L}}{\xi}{}_i^\lambda + g_{\lambda\mu}\partial_\nu \overset{\mathsf{L}}{\xi}{}_i^\lambda = 0. \qquad (1.4.32)$$

In terms of covariant derivatives we have

$$\mathcal{L}_{\overset{\mathsf{L}}{\xi}_i} g_{\mu\nu} \equiv \overset{\mathsf{L}}{\xi}{}_i^\lambda \nabla_\lambda g_{\mu\nu} + g_{\lambda\nu}\nabla_\mu \overset{\mathsf{L}}{\xi}{}_i^\lambda + g_{\lambda\mu}\nabla_\nu \overset{\mathsf{L}}{\xi}{}_i^\lambda = 0, \qquad (1.4.33)$$

$$\nabla^\mu \overset{\mathsf{L}}{\xi}{}_i^\nu + \nabla^\nu \overset{\mathsf{L}}{\xi}{}_i^\mu = 0. \qquad (1.4.34)$$

Using

$$\overset{\mathsf{L}}{\omega}{}_\mu^i = g_{\mu\nu}\delta_{ij} \overset{\mathsf{L}}{\xi}{}_j^\nu, \qquad (1.4.35)$$

the Killing equation may be written for forms

$$\partial_\mu \overset{\mathsf{L}}{\omega}{}_\nu^i + \partial_\nu \overset{\mathsf{L}}{\omega}{}_\mu^i - 2\, \overset{\mathsf{L}}{\omega}{}_\lambda^i \Gamma^\lambda_{\mu\nu} = 0, \qquad (1.4.36)$$

or in terms of covariant derivatives (compare with (1.4.27))

$$\nabla_\mu \overset{\mathsf{L}}{\omega}{}_\nu^i + \nabla_\nu \overset{\mathsf{L}}{\omega}{}_\mu^i = 0. \qquad (1.4.37)$$

So $\overset{\mathsf{L}}{\omega}$ satisfy "transversality" condition

$$\nabla^\mu \overset{\mathsf{L}}{\omega}{}_\mu^i = 0. \qquad (1.4.38)$$

1.5 The space of external differential forms

For the Maurer-Cartan form

$$\overset{\mathsf{L}}{\omega} = \frac{1}{2}\begin{pmatrix} -i(d\gamma + d\alpha\, \cos\beta) & -\mathbf{e}^{i\gamma}(d\beta - i\, d\alpha\, \sin\beta) \\ \mathbf{e}^{-i\gamma}(d\beta + i\, d\alpha\, \sin\beta) & i(d\gamma + d\alpha\, \cos\beta) \end{pmatrix} \qquad (1.5.1)$$

it is possible to construct exterior products of two and three forms. Let us discuss exterior product of two forms

$$\overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega} = \tau_{[i}\,\tau_{j]}\, \overset{\mathsf{L}}{\omega}{}^i \wedge \overset{\mathsf{L}}{\omega}{}^j = \overset{\mathsf{L}}{\Omega}_{[\mu}\, \overset{\mathsf{L}}{\Omega}_{\nu]}\, d\alpha^\mu \wedge d\alpha^\nu. \qquad (1.5.2)$$

The alternation with respect to permutations of indices is

$$T_{[i_1 \dots i_k]} = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) T_{\sigma(i_1,\dots,i_k)}, \qquad (1.5.3)$$

where the sum is taken over all possible permutations of indices. In the space of external differential forms we introduce the following basis

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) dx^{\sigma(i_1} \otimes \cdots \otimes dx^{i_k)}. \qquad (1.5.4)$$

In general, the space of external differential forms is a direct sum of spaces of external differential forms

$$\bigwedge = \bigoplus_{i=0}^{n} \overset{i}{\bigwedge}. \tag{1.5.5}$$

The dimension of this space is

$$\dim \bigwedge = 2^n \tag{1.5.6}$$

and dimension of space of i-forms is

$$\dim \overset{i}{\bigwedge} = C_i^n = \frac{n!}{i!(n-i)!} \tag{1.5.7}$$

In particular, for $SU(2)$ group we have $\dim \bigwedge = 2^3 = 8$ and

$$\begin{aligned} \dim \overset{0}{\bigwedge} &= \dim \overset{3}{\bigwedge} = 1 \\ \dim \overset{1}{\bigwedge} &= \dim \overset{2}{\bigwedge} = 3. \end{aligned} \tag{1.5.8}$$

0-forms are functions on $SU(2)$ manifolds, Maurer-Cartan forms (1.1.12) serve as the basis of 1-forms. Using the expressions for commutators we write the product of two 1-forms (1.5.2)

$$\overset{L}{\omega} \wedge \overset{L}{\omega} = -\sqrt{|g|}\, \epsilon_{\mu\nu\lambda}\, g^{\lambda\rho}\, \overset{L}{\Omega}_\rho\, d\alpha^\mu \wedge d\alpha^\nu, \qquad \mu < \nu \tag{1.5.9}$$

where $\sqrt{|g|} = -\det(\omega) = \sin\beta$, or in components $\overset{L}{\omega}{}^i_\mu$

$$\overset{L}{\omega} \wedge \overset{L}{\omega} = \epsilon_{ijk}\, \tau_k\, \overset{L}{\omega}{}^i_\mu\, \overset{L}{\omega}{}^j_\nu\, d\alpha^\mu \wedge d\alpha^\nu, \qquad \mu < \nu, \tag{1.5.10}$$

$$\begin{aligned} &\left\{ \begin{array}{l} \mu = 1 \\ \nu = 2 \end{array} \right. \quad \epsilon_{ijk}\, \tau_k\, \overset{L}{\omega}{}^i_1\, \overset{L}{\omega}{}^j_2 = \cos\beta\,(-\cos\gamma\,\tau_1 + \sin\gamma\,\tau_2) - \sin\beta\,\tau_3 \\ &\left\{ \begin{array}{l} \mu = 1 \\ \nu = 3 \end{array} \right. \quad \epsilon_{ijk}\, \tau_k\, \overset{L}{\omega}{}^i_1\, \overset{L}{\omega}{}^j_3 = \sin\beta\,(\sin\gamma\,\tau_1 + \cos\gamma\,\tau_2) \\ &\left\{ \begin{array}{l} \mu = 2 \\ \nu = 3 \end{array} \right. \quad \epsilon_{ijk}\, \tau_k\, \overset{L}{\omega}{}^i_2\, \overset{L}{\omega}{}^j_3 = \cos\gamma\,\tau_1 - \sin\gamma\,\tau_2 \end{aligned} \tag{1.5.11}$$

$$2i\, \overset{L}{\omega} \wedge \overset{L}{\omega} = -\begin{pmatrix} \sin\beta & \cos\beta\, \mathbf{e}^{i\gamma} \\ \cos\beta\, \mathbf{e}^{-i\gamma} & -\sin\beta \end{pmatrix} d\alpha^1 \wedge d\alpha^2 \tag{1.5.12}$$

$$+i \begin{pmatrix} 0 & -\sin\beta\, \mathbf{e}^{i\gamma} \\ \sin\beta\, \mathbf{e}^{-i\gamma} & 0 \end{pmatrix} d\alpha^1 \wedge d\alpha^3 \tag{1.5.13}$$

$$+ \begin{pmatrix} 0 & \mathbf{e}^{i\gamma} \\ \mathbf{e}^{-i\gamma} & 0 \end{pmatrix} d\alpha^2 \wedge d\alpha^3. \tag{1.5.14}$$

$$\begin{aligned} \overset{L}{\omega} \wedge \overset{L}{\omega} =& \tau_1(-\cos\beta\cos\gamma\, d\alpha \wedge d\beta + \sin\beta\sin\gamma\, d\alpha \wedge d\gamma + \cos\gamma\, d\beta \wedge d\gamma) \\ &+ \tau_2(\cos\beta\sin\gamma\, d\alpha \wedge d\beta + \cos\gamma\sin\beta\, d\alpha \wedge d\gamma - \sin\gamma\, d\beta \wedge d\gamma) \\ &+ \tau_3(-\sin\beta\, d\alpha \wedge d\beta). \end{aligned} \tag{1.5.15}$$

The exterior product of three forms is

$$\overset{L}{\omega} \wedge \overset{L}{\omega} \wedge \overset{L}{\omega} = \tau_{[i}\, \tau_j\, \tau_{k]}\, \overset{L}{\omega}{}^i \wedge \overset{L}{\omega}{}^j \wedge \overset{L}{\omega}{}^k = -\frac{1}{8}\, \epsilon_{ijk}\, I_2\, \overset{L}{\omega}{}^i \wedge \overset{L}{\omega}{}^j \wedge \overset{L}{\omega}{}^k \tag{1.5.16}$$

Using

$$\epsilon_{ijk}\, \overset{L}{\omega}{}^i_{\mu}\, \overset{L}{\omega}{}^j_{\nu}\, \overset{L}{\omega}{}^k_{\lambda} = \det(\omega)\, \epsilon_{\mu\nu\lambda} \tag{1.5.17}$$

we write

$$\overset{L}{\omega} \wedge \overset{L}{\omega} \wedge \overset{L}{\omega} = \frac{3}{4}\, I_2\, dV \tag{1.5.18}$$

where the invariant volume element is introduced

$$dV := \sqrt{|g|}\, d\alpha \wedge d\beta \wedge d\gamma. \tag{1.5.19}$$

Let us define a duality operator denoted by $*$ (the Hodge star) which establishes an isomorphism of spaces of external differential forms $\bigwedge^k \to \bigwedge^{n-k}$ by $\lambda \mapsto *\lambda$. An k-form λ

$$\lambda = \frac{1}{k!}\, \lambda_{i_1 \ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \tag{1.5.20}$$

has a dual $(n-k)$-form $*\,\lambda$

$$*\,\lambda = \frac{\sqrt{|g|}}{(n-k)! \cdot k!}\, \lambda^{i_1 \ldots i_k}\, \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n}\, dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}, \tag{1.5.21}$$

where $\epsilon_{i_1 \ldots i_n}$ is totally antisymmetric unit tensor (see appendix A). Tensor indices are raised by metric

$$\lambda^{i_1 \ldots i_k} = g^{i_1 j_1} \cdots g^{i_k j_k}\, \lambda_{j_1 \ldots j_k}. \tag{1.5.22}$$

If we have some other k-form η

$$\eta = \frac{1}{k!}\, \eta_{i_1 \ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \tag{1.5.23}$$

then

$$\lambda \wedge *\eta = \frac{1}{k!}\, \lambda_{i_1 \ldots i_k}\, \eta^{i_1 \ldots i_k}\, dV, \qquad dV = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n. \tag{1.5.24}$$

The star operator has the following properties

$$*\,*\,\omega_k = \operatorname{sgn}(g)\, (-)^{k(n-k)}\, \omega_k \tag{1.5.25}$$

$$*1 = dV, \qquad *dV = \operatorname{sgn}(g)\, 1. \tag{1.5.26}$$

The last equalities connect the one dimensional bases of 0-forms and n-forms on the n-dimensional manifold. For $SU(2)$ we have

$$*\, \overset{L}{\omega} = -\, \overset{L}{\omega} \wedge \overset{L}{\omega} \tag{1.5.27}$$

$$*(\overset{L}{\omega} \wedge \overset{L}{\omega}) = -\, \overset{L}{\omega} \tag{1.5.28}$$

$$*(\overset{L}{\omega} \wedge \overset{L}{\omega} \wedge \overset{L}{\omega}) = \frac{3}{4}\, I_2 \tag{1.5.29}$$

1.6 The exterior derivative and the Maurer-Cartan equation

Let us have a k-form ω on n-dimensional orientable Riemannian manifold

$$\omega = \sum_{i_1<i_2<\cdots<i_k} F_{i_1 i_2 \dots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{k!} F_{i_1 \dots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{1.6.1}$$

where $F_{i_1 \dots i_k}$ is an antisymmetric tensor of the rank k.

The exterior derivative of the ω

$$d\,\omega := \sum_{i_1<i_2<\cdots<i_k} \frac{\partial\, F_{i_1 i_2 \dots i_k}}{\partial\, x^{i_0}}\, dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{1.6.2}$$

in other form with free indices

$$d\,\omega = \frac{1}{k!} \frac{\partial\, F_{i_1 i_2 \dots i_k}}{\partial\, x^{i_0}}\, dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{1.6.3}$$

Antisymmetrization of $\partial F_{i_1 i_2 \dots i_k}/\partial x^{i_0}$

$$d\,\omega = \frac{1}{(k+1)!} \varphi_{i_0 i_1 \dots i_k}\, dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{1.6.4}$$

where

$$\varphi_{i_0 \dots i_k} = \sum_{r=0}^{k} (-)^r \frac{\partial}{\partial\, x^{i_r}} F_{i_0 \dots i_{r-1} i_{r+1} \dots i_k}. \tag{1.6.5}$$

$$d\,(\omega_1 \wedge \omega_2) = d\,\omega_1 \wedge \omega_2 + (-)^{\deg \omega_1}\, \omega_1 \wedge d\,\omega_2 \tag{1.6.6}$$

The Maurer-Cartan equation

$$d \overset{\mathsf{L}}{\omega} + \overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega} = 0, \qquad d \overset{\mathsf{R}}{\omega} - \overset{\mathsf{R}}{\omega} \wedge \overset{\mathsf{R}}{\omega} = 0, \tag{1.6.7}$$

may be written in orthonormal basis:

$$d \overset{\mathsf{L}}{\omega}{}^k + \frac{1}{2} \epsilon_{ijk} \overset{\mathsf{L}}{\omega}{}^i \wedge \overset{\mathsf{L}}{\omega}{}^j = 0, \qquad d \overset{\mathsf{R}}{\omega}{}^k - \frac{1}{2} \epsilon_{ijk} \overset{\mathsf{R}}{\omega}{}^i \wedge \overset{\mathsf{R}}{\omega}{}^j = 0, \tag{1.6.8}$$

or in natural basis using ω-matrices:

$$\partial_\mu \overset{\mathsf{L}}{\omega}{}^k_\nu - \partial_\nu \overset{\mathsf{L}}{\omega}{}^k_\mu + \epsilon_{mnk} \overset{\mathsf{L}}{\omega}{}^m_\mu \overset{\mathsf{L}}{\omega}{}^n_\nu = 0, \qquad \partial_\mu \overset{\mathsf{R}}{\omega}{}^k_\nu - \partial_\nu \overset{\mathsf{R}}{\omega}{}^k_\mu - \epsilon_{mnk} \overset{\mathsf{R}}{\omega}{}^m_\mu \overset{\mathsf{R}}{\omega}{}^n_\nu = 0, \tag{1.6.9}$$

and using Ω-matrices:

$$\partial_\mu \overset{\mathsf{L}}{\Omega}_\nu - \partial_\nu \overset{\mathsf{L}}{\Omega}_\mu - \sqrt{|g|}\, \epsilon_{\mu\nu\lambda}\, g^{\lambda\rho} \overset{\mathsf{L}}{\Omega}_\rho = 0, \tag{1.6.10}$$

$$\partial_\mu \overset{\mathsf{R}}{\Omega}_\nu - \partial_\nu \overset{\mathsf{R}}{\Omega}_\mu + \sqrt{|g|}\, \epsilon_{\mu\nu\lambda}\, g^{\lambda\rho} \overset{\mathsf{R}}{\Omega}_\rho = 0. \tag{1.6.11}$$

We may substitute in them components of $\overset{\mathsf{L}}{\omega}$ or $\overset{\mathsf{L}}{\Omega}$ to check derived values of components. From other hand, one can directly apply the exterior derivative d to ω (1.5.1) in matrix form and compare the result with $\overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega}$ (1.5.2). Repeating derivation leads to zero according to the properties $d\,d\,\omega = 0$. For 2-form and 3-form we have

$$d(\overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega}) = d(\overset{\mathsf{R}}{\omega} \wedge \overset{\mathsf{R}}{\omega}) = 0, \tag{1.6.12}$$

$$d(\overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega} \wedge \overset{\mathsf{L}}{\omega}) = d(\overset{\mathsf{R}}{\omega} \wedge \overset{\mathsf{R}}{\omega} \wedge \overset{\mathsf{R}}{\omega}) = 0. \tag{1.6.13}$$

1.7 De Rham operator, Laplacian and Casimir C_2 operators

As for a form $\omega \in \bigwedge^k$ we have

$$* * \omega = \mathrm{sgn}\,(g)\,(-)^{k(n-k)}\,\omega \tag{1.7.1}$$

we may introduce the inverse of Hodge star operator

$$*^{-1} := \mathrm{sgn}(g)(-1)^{k(n-k)} * \qquad \Longrightarrow \qquad *^{-1} * \omega = \omega. \tag{1.7.2}$$

We define the adjoint of the exterior derivative

$$\delta\omega := (-1)^k *^{-1}\; d * \omega \tag{1.7.3}$$

where operator $*^{-1}$ acts on the form $(d * \omega) \in \bigwedge^{n-k+1}$. After simplifications we derive

$$\delta\omega = \mathrm{sgn}(g)(-)^{nk+n+1}\; * \; d * \omega. \tag{1.7.4}$$

δ reduces the degree of a differential form by one unit, whereas d increases the degree :

$$d : \quad \overset{k}{\bigwedge} \longrightarrow \overset{k+1}{\bigwedge} \tag{1.7.5}$$

$$\delta : \quad \overset{k}{\bigwedge} \longrightarrow \overset{k-1}{\bigwedge} \tag{1.7.6}$$

For compact manifolds $\mathcal{M}$ operators d and δ are the metric transposes of each other

$$(d\alpha, \beta) = \int_{\mathcal{M}} d\alpha \wedge *\beta = \int_{\mathcal{M}} \alpha \wedge *\delta\beta = (\alpha, \delta\beta) \tag{1.7.7}$$

where

$$\alpha \in \overset{k-1}{\bigwedge}, \qquad \beta \in \overset{k}{\bigwedge}. \tag{1.7.8}$$

For $SU(2)$ we have

$$\delta\, 1 = 0, \qquad \delta\,(\overset{L}{\omega} \wedge \overset{L}{\omega}) = -\overset{L}{\omega}, \tag{1.7.9}$$

$$\delta\, \overset{L}{\omega} = 0, \qquad \delta\,(\overset{L}{\omega} \wedge \overset{L}{\omega} \wedge \overset{L}{\omega}) = 0. \tag{1.7.10}$$

The De Rham operator is

$$\Delta = (d + \delta)^2 = d\,\delta + \delta\, d \tag{1.7.11}$$

For $SU(2)$ we have

$$\Delta\, \overset{L}{\omega} = \overset{L}{\omega}, \tag{1.7.12}$$

$$\Delta\,(\overset{L}{\omega} \wedge \overset{L}{\omega}) = \overset{L}{\omega} \wedge \overset{L}{\omega}, \tag{1.7.13}$$

$$\Delta\,(\overset{L}{\omega} \wedge \overset{L}{\omega} \wedge \overset{L}{\omega}) = 0. \tag{1.7.14}$$

On the Riemannian space operators d, δ, Δ may be expressed via covariant derivative ∇ in the following way

$$(d\omega)_{i_0 i_1 \ldots i_k} = \nabla_{[i_0}\omega_{i_1 \ldots i_k]} \tag{1.7.15}$$

$$(\delta\omega)_{i_1 \ldots i_{k-1}} = -\nabla^j \alpha_{j i_1 \ldots i_{k-1}} \tag{1.7.16}$$

$$(\Delta\omega)_{i_1 \ldots i_k} = -\nabla^j\nabla_j\omega_{i_1 \ldots i_k} + \sum_{\nu=1}^{k}(-1)^\nu(\nabla_{i_\nu}\nabla^j - \nabla^j\nabla_{i_\nu})\,\omega_{j i_1 \ldots \hat{i}_\nu \ldots i_k} \tag{1.7.17}$$

where $\hat{i}_\nu$ means that i_ν is omitted. The commutator of covariant derivatives may be expressed through the curvature tensor using Ricci identities

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\,\omega_{i_1\ldots i_k} = \omega_{i_1\ldots i_{j-1}\lambda i_{j+1}\ldots i_k}\,\mathcal{R}^{\lambda}_{i_j\nu\mu}. \tag{1.7.18}$$

Using de Rham operator in the form (1.7.17) we can re-derive formulas (1.7.12)-(1.7.14)

$$(\Delta\omega)_\mu = -\nabla^j\nabla_j\,\omega_\mu + \frac{1}{2}\omega_\mu = \omega_\mu \tag{1.7.19}$$

$$\Big(\Delta(\omega\wedge\omega)\Big)_{\mu\nu} = -\nabla^j\nabla_j\,(\omega\wedge\omega)_{\mu\nu} + \frac{1}{2}\,(\omega\wedge\omega)_{\mu\nu} = \,(\omega\wedge\omega)_{\mu\nu} \tag{1.7.20}$$

$$\Big(\Delta(\omega\wedge\omega\wedge\omega)\Big)_{\mu\nu\lambda} = -\nabla^j\nabla_j\,(\omega\wedge\omega\wedge\omega)_{\mu\nu\lambda} = 0 \tag{1.7.21}$$

$$(\omega\wedge\omega\wedge\omega)_{\mu\nu\lambda} = -\sin\beta\epsilon_{\mu\nu\lambda} \tag{1.7.22}$$

For sufficiently well-behaved forms, ω is harmonic if and only if ω is closed and co-closed :

$$\Delta\,\omega = 0 \qquad \Longleftrightarrow \qquad (\,d\,\omega = 0 \quad \& \quad \delta\omega = 0). \tag{1.7.23}$$

On $SU(2)$ group manifold only 3-form is harmonic according to (1.7.14) or, equivalently, (1.7.21). All differential operation may be expressed in terms of covariant derivatives on a Riemannian manifold. Hence, the harmonicity of 3-form follows from the fact that its covariant derivative is zero

$$\nabla_\sigma(\omega\wedge\omega\wedge\omega)_{\mu\nu\lambda} = 0. \tag{1.7.24}$$

The Hodge theorem states that harmonic forms on a compact connected Lie group are just bi-invariant forms ([8], chapter II, §7). We may check that Lie derivative (1.3.1) of 3-form with respect to left/right invariant vector fields is zero.

On the space of scalar functions second term (the sum) in (1.7.17) disappears and de Rham operator converts into the Laplace operator (we omit minus sign)

$$\Delta = g^{\mu\nu}\nabla_\mu\partial_\nu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\right) = g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\right)\partial_\nu, \tag{1.7.25}$$

which coincides with the Casimir operator

$$C_2 = \sum_i \overset{\mathsf{L}}{\xi}_i\,\overset{\mathsf{L}}{\xi}_i = \sum_i \overset{\mathsf{R}}{\xi}_i\,\overset{\mathsf{R}}{\xi}_i, \tag{1.7.26}$$

$$C_2 = \sum_i \left(\overset{\mathsf{L}}{\xi}{}^\mu_i\,\overset{\mathsf{L}}{\xi}{}^\nu_i\partial_\mu\partial_\nu + \overset{\mathsf{L}}{\xi}{}^\mu_i(\partial_\mu\,\overset{\mathsf{L}}{\xi}{}^\nu_i)\partial_\nu\right). \tag{1.7.27}$$

Explicit form for these operators is

$$\Delta = C_2 = \frac{1}{\sin^2\beta}\left(\partial_1^2 + \sin^2\beta\partial_2^2 + \partial_3^2 + \cos\beta(\sin\beta\partial_2 - 2\partial_1\partial_3)\right). \tag{1.7.28}$$

From definition of vector fields $U\tau_i = \overset{\mathsf{L}}{\xi}_i U$ and properties of Pauli matrices (see appendix A) it follows that the Casimir operator C_2 satisfies the equality

$$C_2 U = \sum_i \overset{\mathsf{L}}{\xi}_i\,\overset{\mathsf{L}}{\xi}_i U = \sum_i U\tau_i\tau_i = -\frac{3}{4}U. \tag{1.7.29}$$

Chapter 2

SU(3) group

2.1 Group element

The decomposition of an arbitrary element of SU(3) in Euler angles parametrization may be done in the following way

$$U_\tau = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi}, \tag{2.1.1}$$

using Gell-Mann matrices $\lambda_k = (2i)\tau_k$. Its Hermitian conjugated matrix $U^\dagger$ is

$$U^\dagger = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha}. \tag{2.1.2}$$

In the definition of the left/right invariant forms

$$\overset{\mathsf{L}}{\omega} = U^{-1}dU = \tau_i \overset{\mathsf{L}}{\omega}{}^i \qquad \overset{\mathsf{R}}{\omega} = dUU^{-1} = \tau_i \overset{\mathsf{R}}{\omega}{}^i \tag{2.1.3}$$

the differential of the U matrix can be expressed via differentials of eight Euler parameters

$$dU = \partial_\mu U d\alpha^\mu \qquad \partial_\mu U \equiv \frac{\partial U}{\partial \alpha^\mu}, \tag{2.1.4}$$

where

α^μ	α	β	γ	θ	a	b	c	ϕ
μ	1	2	3	4	5	6	7	8

(2.1.5)

We define Ω_μ matrices through equations

$$\partial_\mu U = U \overset{\mathsf{L}}{\Omega}_\mu \qquad \partial_\mu U = \overset{\mathsf{R}}{\Omega}_\mu U. \tag{2.1.6}$$

$$\overset{\mathsf{L}}{\Omega}_8 = \tau_8 \tag{2.1.7}$$

$$\overset{\mathsf{L}}{\Omega}_7 = \mathbf{e}^{-\tau_8\phi}\,\tau_3\,\mathbf{e}^{\tau_8\phi} \tag{2.1.8}$$

$$\overset{\mathsf{L}}{\Omega}_6 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\,\tau_2\,\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.9}$$

$$\overset{\mathsf{L}}{\Omega}_5 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\,\tau_3\,\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.10}$$

$$\overset{\mathsf{L}}{\Omega}_4 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\,\tau_5\,\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.11}$$

$$\overset{\mathsf{L}}{\Omega}_3 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\,\tau_3\,\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.12}$$

$$\overset{\mathsf{L}}{\Omega}_2 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\,\tau_2\,\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.13}$$

$$\overset{\mathsf{L}}{\Omega}_1 = \mathbf{e}^{-\tau_8\phi}\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\,\tau_3\,\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\mathbf{e}^{\tau_8\phi} \tag{2.1.14}$$

$$\overset{\mathsf{R}}{\Omega}_1 = \tau_3 \tag{2.1.15}$$
$$\overset{\mathsf{R}}{\Omega}_2 = \mathbf{e}^{\tau_3\alpha}\,\tau_2\,\mathbf{e}^{-\tau_3\alpha} \tag{2.1.16}$$
$$\overset{\mathsf{R}}{\Omega}_3 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\,\tau_3\,\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.17}$$
$$\overset{\mathsf{R}}{\Omega}_4 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\,\tau_5\,\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.18}$$
$$\overset{\mathsf{R}}{\Omega}_5 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\,\tau_3\,\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.19}$$
$$\overset{\mathsf{R}}{\Omega}_6 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\,\mathbf{e}^{\tau_3 a}\,\tau_2\,\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.20}$$
$$\overset{\mathsf{R}}{\Omega}_7 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\,\tau_3\,\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.21}$$
$$\overset{\mathsf{R}}{\Omega}_8 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma}\mathbf{e}^{\tau_5\theta}\mathbf{e}^{\tau_3 a}\mathbf{e}^{\tau_2 b}\mathbf{e}^{\tau_3 c}\,\tau_8\,\mathbf{e}^{-\tau_3 c}\mathbf{e}^{-\tau_2 b}\mathbf{e}^{-\tau_3 a}\mathbf{e}^{-\tau_5\theta}\mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\alpha} \tag{2.1.22}$$

We substitute these matrices into the formulas for components of the basis forms

$$\overset{\mathsf{L}}{\omega}{}^i_\mu = -2\mathrm{Tr}(\tau_i \overset{\mathsf{L}}{\Omega}_\mu), \qquad \overset{\mathsf{R}}{\omega}{}^i_\mu = -2\mathrm{Tr}(\tau_i \overset{\mathsf{R}}{\Omega}_\mu). \tag{2.1.23}$$

$$\overset{\mathsf{L}}{\omega}{}^i = \overset{\mathsf{L}}{\omega}{}^i_\mu d\alpha^\mu, \qquad \overset{\mathsf{R}}{\omega}{}^i = \overset{\mathsf{R}}{\omega}{}^i_\mu d\alpha^\mu. \tag{2.1.24}$$

Calculations of vector fields and the adjoint matrix $SO(8)$ may be done according to section 1.2 where matrices τ are understood as $\tau = \lambda/(2i)$.

2.2 Riemannian structure on SU(3) manifold

Using formulas from the section 1.4 we get the following results.

- Metric tensor

$$\begin{aligned}
g_{down_{SU(3)}} \;=\;& d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta\, d\alpha\, d\gamma && (2.2.1)\\
& + da^2 + db^2 + dc^2 + 2\cos b\, da\, dc && (2.2.2)\\
& + d\theta^2 + d\phi^2 - \sqrt{3}\sin^2\frac{\theta}{2}(\cos\beta\, d\alpha + d\gamma)d\phi && (2.2.3)\\
& + (1+\cos^2\frac{\theta}{2})(\cos\beta\, d\alpha + d\gamma)(da + \cos b\, dc) && (2.2.4)\\
& + 2\cos\frac{\theta}{2}\sin(a+\gamma)(\sin\beta\, d\alpha\, db + \sin b\, d\beta\, dc) && (2.2.5)\\
& + 2\cos\frac{\theta}{2}\cos(a+\gamma)(d\beta\, db - \sin\beta\sin b\, d\alpha\, dc) && (2.2.6)
\end{aligned}$$

- Inverse metric tensor

$$g^{up}_{SU(3)} = \frac{1}{\sin^2\frac{\theta}{2}}\left(\frac{1}{\sin^2\beta}(\partial_1^2+\sin^2\beta\,\partial_2^2+\partial_3^2-2\cos\beta\,\partial_1\,\partial_3)\right. \tag{2.2.7}$$

$$\left.+\frac{1}{\sin^2 b}(\partial_5^2+\sin^2 b\,\partial_6^2+\partial_7^2-2\cos b\,\partial_5\,\partial_7)\right) \tag{2.2.8}$$

$$+2\frac{\cot\frac{\theta}{2}}{\sin\frac{\theta}{2}}\left(\cos(a+\gamma)\Big(\frac{1}{\sin\beta\sin b}(\,\partial_1-\cos\beta\,\partial_3)(\,\partial_7-\cos b\,\partial_5)-\,\partial_2\,\partial_6\Big)\right. \tag{2.2.9}$$

$$\left.-\sin(a+\gamma)\Big(\frac{1}{\sin\beta}(\,\partial_1-\cos\beta\,\partial_3)\,\partial_6+\frac{1}{\sin b}(\,\partial_7-\cos b\,\partial_5)\,\partial_2\Big)\right) \tag{2.2.10}$$

$$-\frac{2}{\sin^2\frac{\theta}{2}}\,\partial_3\,\partial_5+\partial_4^2+\frac{1}{4}(\sqrt{3}\partial_5-\partial_8)^2 \tag{2.2.11}$$

$$+\frac{1}{\cos^2\frac{\theta}{2}}\Big((\partial_3-\partial_5)(\partial_3+\sqrt{3}\,\partial_8)+\frac{1}{4}(\partial_5+\sqrt{3}\,\partial_8)^2\Big) \tag{2.2.12}$$

- Determinants of metric and forms

$$\det g_{down_{SU(3)}} = (\det g^{up}_{SU(3)})^{-1} = \cos^2\frac{\theta}{2}\sin^6\frac{\theta}{2}\sin^2\beta\sin^2 b \tag{2.2.13}$$

$$\det\omega_{SU(3)} = \cos\frac{\theta}{2}\sin^3\frac{\theta}{2}\sin\beta\sin b \tag{2.2.14}$$

- Ricci tensor

$$\mathcal{R}_{\mu\nu} = \frac{3}{4}\,g_{\mu\nu}, \qquad \det\mathcal{R}_{\mu\nu} = \left(\frac{3}{4}\right)^8\det(g_{\mu\nu}) \tag{2.2.15}$$

- Curvature scalar

$$\mathcal{R} = \frac{3}{4}g^{\mu\nu}g_{\mu\nu} = \frac{3}{4}\,8 = 6, \qquad g^{\mu\nu}g_{\mu\nu} = n = 8 \tag{2.2.16}$$

2.3 Casimir operators and Laplacian

$SU(3)$ group has two Casimir operators: second order operator C_2 and third order operator C_3. The operator C_2

$$C_2 = \overset{L}{\xi}_i\,\overset{L}{\xi}_i = (\overset{L}{\xi}{}^{\mu}_i\,\overset{L}{\xi}{}^{\nu}_i)\partial_\mu\partial_\nu + \overset{L}{\xi}{}^{\mu}_i(\partial_\mu\,\overset{L}{\xi}{}^{\nu}_i)\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu + \overset{L}{\xi}{}^{\mu}_i(\partial_\mu\,\overset{L}{\xi}{}^{\nu}_i)\partial_\nu \tag{2.3.17}$$

coinsides with the laplacian and has the form

$$C_2 = g^{up}_{SU(3)} + \frac{1}{\sin^2\frac{\theta}{2}}(\cot\beta\,\partial_2+\cot b\,\partial_6+\partial_4)+\cot\theta\,\partial_4. \tag{2.3.18}$$

From definition of the left invariant vector fields $\overset{L}{\xi}_i U = U\tau_i$ and properties of Gell-Mann matrices (see appendix C) it follows that C_2 satisfies equality

$$C_2 U = U\tau_i\tau_i = -\frac{4}{3}U. \tag{2.3.19}$$

Third order Casimir operator is

$$C_3 = d_{ijk} \overset{L}{\xi}_i \overset{L}{\xi}_j \overset{L}{\xi}_k = d_{ijk}(\overset{L}{\xi}{}^\mu_i \partial_\mu)(\overset{L}{\xi}{}^\nu_j \partial_\nu)(\overset{L}{\xi}{}^\lambda_k \partial_\lambda) \tag{2.3.20}$$

or

$$\begin{aligned} C_3 \;\; = \;\; & d_{ijk} \overset{L}{\xi}{}^\mu_i \overset{L}{\xi}{}^\nu_j \overset{L}{\xi}{}^\lambda_k \, \partial_\mu \partial_\nu \partial_\lambda & (2.3.21)\\ & + d_{ijk} \overset{L}{\xi}{}^\mu_i \Big(\partial_\mu (\overset{L}{\xi}{}^\nu_j \overset{L}{\xi}{}^\lambda_k) + \overset{L}{\xi}{}^\nu_j \partial_\mu (\overset{L}{\xi}{}^\lambda_k) \Big) \, \partial_\nu \partial_\lambda & (2.3.22)\\ & + d_{ijk} \overset{L}{\xi}{}^\mu_i \Big((\partial_\mu \overset{L}{\xi}{}^\nu_j)(\partial_\nu \overset{L}{\xi}{}^\lambda_k) + \overset{L}{\xi}{}^\nu_j (\partial_\mu \partial_\nu \overset{L}{\xi}{}^\lambda_k) \Big) \, \partial_\lambda . & (2.3.23) \end{aligned}$$

C_3 satisfies to formulas

$$C_3 U = U d_{ijk} \tau_i \tau_j \tau_k = i \frac{10}{9} U \tag{2.3.24}$$

and

$$C_3 = i C_2 (2 C_2 + \frac{11}{6} I_3). \tag{2.3.25}$$

Below we give explicit form of C_3 splited into three parts which contain third, second and first derivatives

$$C_3 = 3(\mathcal{A} + \mathcal{B} + \frac{1}{2}\mathcal{C}) \tag{2.3.26}$$

$$\mathcal{A} = \frac{1}{2}\, a_1 + a_2 \cos(a+\gamma) + a_3 \sin(a+\gamma) \tag{2.3.27}$$

$$\begin{aligned} a_1 \;\; = \;\; & \frac{1}{\sin^2 \frac{\theta}{2}} \Big(2(\partial_3 + \frac{1}{\sqrt{3}}\, \partial_8)\, \partial_3 \partial_5 & (2.3.28)\\ & \qquad - \frac{1}{\sin^2 \beta} (\partial_5 + \frac{1}{\sqrt{3}}\, \partial_8)(\partial_1^2 + \sin^2 \beta \, \partial_2^2 + \partial_3^2 - 2 \cos \beta \, \partial_1 \partial_3) & (2.3.29)\\ & \qquad - \frac{1}{\sin^2 b} (2\partial_3 - \partial_5 + \frac{1}{\sqrt{3}}\, \partial_8)(\partial_5^2 + \sin^2 b \, \partial_6^2 + \partial_7^2 - 2 \cos b \, \partial_5 \partial_7) \Big) & (2.3.30)\\ & + \frac{1}{\cos^2 \frac{\theta}{2}} \Big((\partial_3 - \partial_5)(\partial_3 \partial_5 - (\sqrt{3}\, \partial_8 + \partial_3 - \partial_5) \frac{1}{\sqrt{3}} \partial_8) & (2.3.31)\\ & \qquad + (\frac{2}{\sqrt{3}} \partial_3 - \frac{\sqrt{3}}{4}\, \partial_5 + \frac{1}{4} \partial_8)\, \partial_5 \partial_8 - \frac{\sqrt{3}}{4} \partial_8^3 \Big) & (2.3.32)\\ & + (\partial_4^2 + \frac{\sqrt{3}}{4}\, \partial_5 \partial_8)(\partial_5 - \frac{1}{\sqrt{3}} \partial_8) + \frac{1}{4} \tan^2 \frac{\theta}{2} \partial_5^3 + \frac{1}{4} \left(\frac{\partial_8}{\sqrt{3}} \right)^3 & (2.3.33) \end{aligned}$$

$$\begin{aligned} a_2 \;\; = \;\; & \frac{1}{\sin \frac{\theta}{2}} \left(\frac{1}{\sin \beta} (\partial_1 - \cos \beta \, \partial_3)\, \partial_6 + \frac{1}{\sin b} (\partial_7 - \cos b \, \partial_5)\, \partial_2 \right) \partial_4 & (2.3.34)\\ & + \frac{1}{\cos \frac{\theta}{2}} \left(-\frac{1}{2} \partial_5 + \frac{1}{\sin^2 \frac{\theta}{2}} \left(\partial_3 + \frac{1}{4\sqrt{3}} (5 - \cos \theta) \partial_8 \right) \right) \times & (2.3.35)\\ & \qquad \times \left(\partial_2 \partial_6 - \frac{1}{\sin \beta \sin b} (\partial_1 - \cos \beta \, \partial_3)(\partial_7 - \cos b \, \partial_5) \right) & (2.3.36) \end{aligned}$$

$$a_3 = \frac{1}{\cos\frac{\theta}{2}}\left(-\frac{1}{2}\partial_5 + \frac{1}{\sin^2\frac{\theta}{2}}\left(\partial_3 + \frac{1}{4\sqrt{3}}(5-\cos\theta)\partial_8\right)\right)\times \tag{2.3.37}$$

$$\times\left(\frac{1}{\sin\beta}(\partial_1 - \cos\beta\,\partial_3)\,\partial_6 + \frac{1}{\sin b}(\partial_7 - \cos b\,\partial_5)\,\partial_2\right) \tag{2.3.38}$$

$$-\frac{1}{\sin\frac{\theta}{2}}\left(\partial_2\partial_6 - \frac{1}{\sin\beta\sin b}(\partial_1 - \cos\beta\,\partial_3)(\partial_7 - \cos b\,\partial_5)\right)\,\partial_4 \tag{2.3.39}$$

$$\sin^2\frac{\theta}{2}\,\mathcal{B} = \cos\frac{\theta}{2}\cos(a+\gamma)\Big(\frac{1}{\sin\beta}(\,\partial_1 - \cos\beta\,\partial_3)\,\partial_6 + \frac{1}{\sin b}(\,\partial_7 - \cos b\,\partial_5)\,\partial_2\Big) \tag{2.3.40}$$

$$+\cos\frac{\theta}{2}\sin(a+\gamma)\Big(\frac{1}{\sin\beta\sin b}(\,\partial_1 - \cos\beta\,\partial_3)(\,\partial_7 - \cos b\,\partial_5) - \,\partial_2\,\partial_6\Big) \tag{2.3.41}$$

$$+\frac{1}{2}\Big(\cot\beta\,\partial_2 + \frac{\sin\frac{3}{2}\theta}{2\cos\frac{\theta}{2}}\,\partial_4 + \cot b\,\partial_6\Big)\Big(\partial_5 - \frac{1}{\sqrt{3}}\,\partial_8\Big) \tag{2.3.42}$$

$$-\cot\beta\,\partial_2\partial_5 - \cot b\,\partial_3\partial_6 \tag{2.3.43}$$

$$\mathcal{C} = -\partial_5 + \frac{1}{\sqrt{3}}\,\partial_8 \tag{2.3.44}$$

2.4 Harmonic forms

The Betti numbers b_p [7] are equal to number of harmonic p-forms. In the case of $SU(3)$ they are [3]

$$b_p: \qquad 1,0,0,1,0,1,0,0,1 \qquad p = 0,1,\ldots,8. \tag{2.4.45}$$

The first is 0-form, unity. The last is the form of invariant volume with one significant component given by determinant of left (or right) invariant forms (2.2.14)

$$\Theta_8 = *1 = \sqrt{g}\,d\alpha\wedge d\beta\wedge d\gamma\wedge d\theta\wedge da\wedge db\wedge dc\wedge d\phi. \tag{2.4.46}$$

So we need to compute only 3-form and its dual - 5-form. 3-form is

$$\Theta_3 = \mathrm{Tr}(\omega\wedge\omega\wedge\omega) = \mathrm{Tr}(\tau_{[a}\tau_b\tau_{c]})\,\omega^a\wedge\omega^b\wedge\omega^c, \tag{2.4.47}$$

where $\omega = U^{-1}dU$. Using the trace of antisymmetrized product of three τ matrices

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_{c]}) = -\frac{1}{4}f_{abc}. \tag{2.4.48}$$

we get

$$\Theta_3 = -\frac{1}{4}f_{abc}\,\omega^a\wedge\omega^b\wedge\omega^c = -\frac{1}{4}f_{abc}\,\omega^a_\mu\omega^b_\nu\omega^c_\lambda d\alpha^\mu\wedge d\alpha^\nu\wedge d\alpha^\lambda. \tag{2.4.49}$$

The harmonic 3-form is

$$\Theta_3 = -\frac{3}{2}\psi_{\mu\nu\lambda}\,d\alpha^\mu\wedge d\alpha^\nu\wedge d\alpha^\lambda, \qquad 1\le\mu<\nu<\lambda\le 8 \tag{2.4.50}$$

with coefficients $\psi_{\mu\nu\lambda}$ given in appendix G

$$\psi_{\mu\nu\lambda} := f_{abc}\,\omega^a_\mu\omega^b_\nu\omega^c_\lambda. \tag{2.4.51}$$

5- form is

$$\Theta_5 = \mathrm{Tr}(\omega\wedge\omega\wedge\omega\wedge\omega\wedge\omega) = \mathrm{Tr}(\tau_{[i_1}\dots\tau_{i_5]})\,\omega^{i_1}\wedge\dots\wedge\omega^{i_5}. \tag{2.4.52}$$

The trace in the definition of Θ_5 is

$$\mathrm{Tr}(\tau_{[i_1}\tau_{i_2}\tau_{i_3}\tau_{i_4}\tau_{i_5]}) = \frac{i}{16} f_{[i_1i_2|k}d_{k|i_3|l}f_{l|i_4i_5]} \tag{2.4.53}$$

(alternation with respect to indices $i_1,\dots,i_5$ and sum with respect to indices k,l). With coefficients

$$\chi_{i_1\dots i_5} := \frac{1}{\eta}\mathrm{Tr}(\tau_{[i_1}\tau_{i_2}\tau_{i_3}\tau_{i_4}\tau_{i_5]}), \qquad \eta = \frac{i}{64\sqrt{3}}, \tag{2.4.54}$$

5-form becomes

$$\Theta_5 = \eta\chi_{i_1\dots i_5}\,\omega^{i_1}\wedge\dots\wedge\omega^{i_5} \tag{2.4.55}$$

In holonomic basis the harmonic 5-form looks as it follows

$$\Theta_5 = \eta\Upsilon_{i_1\dots i_5}\,d\alpha^{i_1}\wedge\dots\wedge d\alpha^{i_5}, \tag{2.4.56}$$

where coefficients $\Upsilon_{i_1\dots i_5}$ are introduced (see appendix G)

$$\Upsilon_{i_1\dots i_5} := \chi_{j_1\dots j_5}\,\omega^{j_1}_{i_1}\dots\omega^{j_5}_{i_5}. \tag{2.4.57}$$

Each component of the 3-form has dual component of 5-form. Three indices of a component of the 3-form and five indices of the dual component of 5-form constitute a permutation of the set $\{1,2,\dots,8\}$. In the following table we compare coefficients f_{abc} and χ_{defgh} of dual components of 3-form and 5-form

a	b	c	f_{abc}	d	e	f	g	h	χ_{defgh}	$\mathrm{sgn}(\sigma_{abcdfgh})$
1	2	3	1	4	5	6	7	8	-2	1
1	4	7	1/2	2	3	5	6	8	-1	1
1	5	6	$-1/2$	2	3	4	7	8	1	1
2	4	6	1/2	1	3	5	7	8	-1	1
2	5	7	1/2	1	3	4	6	8	-1	1
3	4	5	1/2	1	2	6	7	8	-1	1
3	6	7	$-1/2$	1	2	4	5	8	1	1
4	5	8	$\sqrt{3}/2$	1	2	3	6	7	$\sqrt{3}$	-1
6	7	8	$\sqrt{3}/2$	1	2	3	4	5	$\sqrt{3}$	-1

(2.4.58)

From

$$f_{[i_1i_2i_3}\chi_{i_4\dots i_8]} = -\frac{1}{7}\epsilon_{i_1\dots i_8}, \tag{2.4.59}$$

$$\psi_{[i_1i_2i_3}\Upsilon_{i_4\dots i_8]} = -\frac{1}{7}\sqrt{g}\epsilon_{i_1\dots i_8} \tag{2.4.60}$$

we see that the product $\Theta_3\wedge\Theta_5$ is proportional to Θ_8

$$\Theta_3\wedge\Theta_5 = \frac{\eta}{28}\epsilon_{i_1\dots i_8}\omega^{i_1}\wedge\dots\wedge\omega^{i_8} = i\frac{15\sqrt{3}}{2}\Theta_8 \tag{2.4.61}$$

Note if we define $\tilde{\Theta}_5 := *\Theta_3$ we shall get the same results up to the some numerical factors.

Appendix A

Pauli matrices

Hermitian matrices $\sigma_i = \sigma_i^\dagger$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

Anti-Hermitian matrices $\tau_i = -\tau_i^\dagger$

$$\tau_k = \frac{\sigma_k}{2i} \tag{A.2}$$

$$\tau_i\,\tau_j = -\frac{1}{4}\,\delta_{ij} I_2 + \frac{1}{2}\,\epsilon_{ijk}\,\tau_k \tag{A.3}$$

Totally antisymmetric unit tensor

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} +1 & \text{even permutation } \sigma \\ -1 & \text{odd permutation } \sigma \\ 0 & \text{any two indices repeated} \end{cases} \tag{A.4}$$

$$[\tau_i, \tau_j] = \epsilon_{ijk}\,\tau_k, \tag{A.5}$$

$$\epsilon_{ijk}\,\epsilon_{mnk} = \delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm} \tag{A.6}$$

$$\tau_i\,\tau_j\,\tau_k = -\frac{1}{8}\,\epsilon_{ijk}\,I_2 - \frac{1}{4}(\delta_{ij}\,\tau_k - \delta_{ik}\,\tau_j + \delta_{jk}\,\tau_i) \tag{A.7}$$

$$\tau_{[i}\,\tau_j\,\tau_{k]} = -\frac{1}{8}\,\epsilon_{ijk}\,I_2 \tag{A.8}$$

Adjoint representation of Lie algebra $\mathfrak{su}(2)$

$$(ad_{\tau_i})_{jk} := -\epsilon_{ijk} \tag{A.9}$$

$$ad_{\tau_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad ad_{\tau_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad ad_{\tau_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.10}$$

$$ad_{\tau_i}\tau_j = \epsilon_{ijk}\tau_k = [\tau_i, \tau_j] \tag{A.11}$$

Cartan metric (the value of the Killing form $\mathcal{K}$ on basis vectors of a Lie algebra)

$$\mathcal{K}(\tau_i, \tau_j) = \mathrm{Tr}(ad_{\tau_i} ad_{\tau_j}) = -2\delta_{ij} \tag{A.12}$$

Appendix B

$SU(2)$ Omega Ω Matrices

$$\overset{\mathsf{L}}{\Omega}_1 = \mathbf{e}^{-\tau_3\gamma}\mathbf{e}^{-\tau_2\beta}\tau_3\mathbf{e}^{\tau_2\beta}\mathbf{e}^{\tau_3\gamma} = \sin\beta(-\cos\gamma\,\tau_1 + \sin\gamma\,\tau_2) + \cos\beta\,\tau_3 \tag{B.1}$$

$$\overset{\mathsf{L}}{\Omega}_1 = \frac{1}{2i}\begin{pmatrix} \cos\beta & -\sin\beta\,\mathbf{e}^{i\gamma} \\ -\sin\beta\,\mathbf{e}^{-i\gamma} & -\cos\beta \end{pmatrix} \tag{B.2}$$

$$\overset{\mathsf{L}}{\Omega}_2 = \mathbf{e}^{-\tau_3\gamma}\tau_2\mathbf{e}^{\tau_3\gamma} = \sin\gamma\,\tau_1 + \cos\gamma\,\tau_2 \tag{B.3}$$

$$\overset{\mathsf{L}}{\Omega}_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathbf{e}^{-i\gamma} \\ \mathbf{e}^{i\gamma} & 0 \end{pmatrix} \tag{B.4}$$

$$\overset{\mathsf{L}}{\Omega}_3 = \overset{\mathsf{R}}{\Omega}_1 = \tau_3 \tag{B.5}$$

$$\overset{\mathsf{R}}{\Omega}_2 = \mathbf{e}^{\tau_3\alpha}\tau_2\mathbf{e}^{-\tau_3\alpha} = -\sin\alpha\,\tau_1 + \cos\alpha\,\tau_2 \tag{B.6}$$

$$\overset{\mathsf{R}}{\Omega}_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathbf{e}^{-i\alpha} \\ \mathbf{e}^{i\alpha} & 0 \end{pmatrix} \tag{B.7}$$

$$\overset{\mathsf{R}}{\Omega}_3 = \mathbf{e}^{\tau_3\alpha}\mathbf{e}^{\tau_2\beta}\tau_3\mathbf{e}^{-\tau_2\beta}\mathbf{e}^{-\tau_3\gamma} = \sin\beta(\cos\alpha\,\tau_1 + \sin\alpha\,\tau_2) + \cos\beta\,\tau_3 \tag{B.8}$$

$$\overset{\mathsf{R}}{\Omega}_3 = \frac{1}{2i}\begin{pmatrix} \cos\beta & \sin\beta\,\mathbf{e}^{-i\alpha} \\ \sin\beta\,\mathbf{e}^{i\alpha} & -\cos\beta \end{pmatrix} \tag{B.9}$$

$$\tau_1 = -\frac{\cos\gamma}{\sin\beta}\left(\overset{\mathsf{L}}{\Omega}_1 - \cos\beta\ \overset{\mathsf{L}}{\Omega}_3\right) + \sin\gamma\ \overset{\mathsf{L}}{\Omega}_2 \tag{B.10}$$

$$\tau_1 = \frac{\cos\alpha}{\sin\beta}\left(\overset{\mathsf{R}}{\Omega}_3 - \cos\beta\ \overset{\mathsf{R}}{\Omega}_1\right) - \sin\alpha\ \overset{\mathsf{R}}{\Omega}_2 \tag{B.11}$$

$$\tau_2 = \frac{\sin\gamma}{\sin\beta}\left(\overset{\mathsf{L}}{\Omega}_1 - \cos\beta\ \overset{\mathsf{L}}{\Omega}_3\right) + \cos\gamma\ \overset{\mathsf{L}}{\Omega}_2 \tag{B.12}$$

$$\tau_2 = \frac{\sin\alpha}{\sin\beta}\left(\overset{\mathsf{R}}{\Omega}_3 - \cos\beta\ \overset{\mathsf{R}}{\Omega}_1\right) + \cos\alpha\ \overset{\mathsf{R}}{\Omega}_2 \tag{B.13}$$

$$\overset{\mathsf{L}}{\Omega}_\mu\ \overset{\mathsf{L}}{\Omega}_\nu = -\frac{1}{4}g_{\mu\nu}\,I_2 - \frac{1}{2}\sqrt{|g|}\,\epsilon_{\mu\nu\lambda}\,g^{\lambda\rho}\ \overset{\mathsf{L}}{\Omega}_\rho \tag{B.14}$$

$$\overset{\mathsf{R}}{\Omega}_\mu \ \overset{\mathsf{R}}{\Omega}_\nu = -\frac{1}{4} g_{\mu\nu}\, I_2 - \frac{1}{2} \sqrt{|g|}\, \epsilon_{\mu\nu\lambda}\, g^{\lambda\rho} \ \overset{\mathsf{R}}{\Omega}_\rho \tag{B.15}$$

$$[\overset{\mathsf{L}}{\Omega}_\mu, \overset{\mathsf{L}}{\Omega}_\nu] = -\epsilon_{\mu\nu\lambda} \sqrt{|g|}\, g^{\lambda\rho} \ \overset{\mathsf{L}}{\Omega}_\rho \tag{B.16}$$

$$[\overset{\mathsf{R}}{\Omega}_\mu, \overset{\mathsf{R}}{\Omega}_\nu] = -\epsilon_{\mu\nu\lambda} \sqrt{|g|}\, g^{\lambda\rho} \ \overset{\mathsf{R}}{\Omega}_\rho \tag{B.17}$$

$$\overset{\mathsf{L}}{\Omega}{}^\dagger_\mu = - \overset{\mathsf{L}}{\Omega}_\mu, \qquad \overset{\mathsf{R}}{\Omega}{}^\dagger_\mu = - \overset{\mathsf{R}}{\Omega}_\mu, \tag{B.18}$$

$$\overset{\mathsf{L}}{\Omega}{}^{-1}_\mu = -4 \ \overset{\mathsf{L}}{\Omega}_\mu, \qquad \overset{\mathsf{R}}{\Omega}{}^{-1}_\mu = -4 \ \overset{\mathsf{R}}{\Omega}_\mu, \tag{B.19}$$

$$\det \ \overset{\mathsf{L}}{\Omega}_\mu = \det \ \overset{\mathsf{R}}{\Omega}_\mu = \frac{1}{4} \tag{B.20}$$

Appendix C

Gell-Mann matrices and $SU(3)$ structure constants

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{C.1}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\tau_k = \frac{\lambda_k}{2i}, \qquad \lambda^\dagger = \lambda, \qquad \tau^\dagger = -\tau \tag{C.2}$$

$$\tau_a \tau_b = -\frac{1}{6}\delta_{ab} I_3 + \frac{1}{2i} d_{abc}\tau_c + \frac{1}{2} f_{abc}\tau_c \tag{C.3}$$

Commutator

$$[\tau_a, \tau_b] = f_{abc}\tau_c \tag{C.4}$$

Anticommutator

$$\{\tau_a, \tau_b\} = -\frac{1}{3}\delta_{ab} I_3 - i d_{abc}\tau_c \tag{C.5}$$

$SU(3)$ structure constants

$$f_{abc} = -2\mathrm{Tr}([\tau_a, \tau_b]\tau_c) \tag{C.6}$$

$$d_{abc} = -2i\mathrm{Tr}(\{\tau_a, \tau_b\}\tau_c) \tag{C.7}$$

$$\begin{aligned} &f_{123} = 1, \\ &f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = 1/2, \\ &f_{458} = f_{678} = \sqrt{3}/2, \end{aligned} \tag{C.8}$$

$$f_{abi}\, f_{dci} = \frac{2}{3}(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) + (d_{adi}\, d_{bci} - d_{aci}\, d_{bdi})\,. \tag{C.9}$$

Jacobi identity

$$f_{abi}f_{icd} + f_{bci}f_{iad} + f_{cai}f_{ibd} = 0 \tag{C.10}$$
$$f_{abi}d_{icd} + f_{bci}d_{iad} + f_{cai}d_{ibd} = -i4!\mathrm{Tr}(\tau_{[a}\tau_b\tau_{c]}\tau_d) \tag{C.11}$$
$$d_{abi}f_{icd} + d_{bci}f_{iad} + d_{cai}f_{ibd} = 0 \tag{C.12}$$
$$d_{abi}d_{icd} + d_{bci}d_{iad} + d_{cai}d_{ibd} = \frac{1}{3}(\delta_{ab}\delta_{cd} + \delta_{bc}\delta_{ad} + \delta_{ca}\delta_{bd}) \tag{C.13}$$

$$f_{abi}d_{icd} = -2i\mathrm{Tr}([\tau_a, \tau_b]\{\tau_c, \tau_d\}) \tag{C.14}$$

$$f_{abc}f_{abd} = 3\delta_{cd} \tag{C.15}$$

$$f_{abc}d_{abd} = 0 \tag{C.16}$$
$$d_{abc}d_{abd} = \frac{5}{3}\delta_{cd} \tag{C.17}$$

$$\sum_a \tau_a\tau_b\tau_a = \frac{1}{6}\tau_b \tag{C.18}$$

$$f_{abc}\tau_b\tau_c = \frac{3}{2}\tau_a \tag{C.19}$$

$$f_{adg}f_{bed}f_{cge} = -\frac{3}{2}f_{abc} \tag{C.20}$$

$$f_{abc}f_{ade}f_{bdf}f_{ceg} = \frac{9}{2}\delta_{fg} \tag{C.21}$$

$SU(3)$ τ matrix products, their alternation and traces

$$\phi_{abc} := \frac{1}{2}(f_{abc} - id_{abc}) \tag{C.22}$$

$$f_{abc} = 2\Re\phi_{abc} = \phi_{abc} + \phi^*_{abc}, \qquad d_{abc} = -2\Im\phi_{abc} = 2\Im\phi^*_{abc} = i(\phi_{abc} - \phi^*_{abc}) \tag{C.23}$$

$$\tau_a\tau_b = -\frac{1}{6}\delta_{ab}I_3 + \phi_{abi}\tau_i \tag{C.24}$$

$$\tau_a\tau_b\tau_c = -\frac{1}{6}(\phi_{abc}I_3 + \delta_{ab}\tau_c) + \phi_{abi}\phi_{icj}\tau_j \tag{C.25}$$

$$\tau_a\tau_b\tau_c\tau_d = -\frac{1}{6}\Big((-\frac{1}{6}\delta_{ab}\delta_{cd} + \phi_{abi}\phi_{icd})I_3 + \phi_{abc}\tau_d + \delta_{ab}\phi_{cdi}\tau_i\Big) + \phi_{abi}\phi_{icj}\phi_{jdf}\tau_f \tag{C.26}$$

$$\begin{aligned}\tau_a\tau_b\tau_c\tau_d\tau_e = &-\frac{1}{6}\Big(-\frac{1}{6}(\delta_{ab}\phi_{cde} + \phi_{abc}\delta_{de})I_3 + \phi_{abi}\phi_{icj}\phi_{jde}I_3 \\ &+ (-\frac{1}{6}\delta_{ab}\delta_{cd} + \phi_{abi}\phi_{icd})\tau_e + \phi_{abc}\phi_{dei}\tau_i + \delta_{ab}\phi_{cdi}\phi_{iej}\tau_j\Big) \\ &+ \phi_{abi}\phi_{icj}\phi_{jdf}\phi_{fek}\tau_k\end{aligned} \tag{C.27}$$

$$\tau_{[i_1}\tau_{i_2}\dots\tau_{i_n]} = A(\tau_{i_1}\tau_{i_2}\dots\tau_{i_n}) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\tau_{\sigma(i_1}\tau_{i_2}\dots\tau_{i_n)} \tag{C.28}$$

$$\tau_{[a}\tau_{b]} = \frac{1}{2}[\tau_a, \tau_b] = \frac{1}{2} f_{abc}\tau_c \tag{C.29}$$

$$\mathrm{Tr}(\tau_{[a}\tau_{b]}) = 0 \tag{C.30}$$

$$\tau_{[a}\tau_b\tau_{c]} = -\frac{1}{12}\Big(f_{abc}I_3 + i(f_{abi}d_{icf} + f_{bci}d_{iaf} + f_{cai}d_{ibf})\tau_f\Big) \tag{C.31}$$

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_{c]}) = -\frac{1}{4} f_{abc} \tag{C.32}$$

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_c\tau_{d]}) = 0 \tag{C.33}$$

$$f_{[ab|i}f_{i|cd]} = 0, \qquad f_{[ab|i}f_{i|c|j}f_{j|de]} = 0 \tag{C.34}$$

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_c\tau_d\tau_{e]}) = \frac{i}{16} f_{[ab|i}d_{i|c|j}f_{j|de]} \tag{C.35}$$

where alternation with respect to indices a, b, c, d, e and summation with respect to indices i, j.

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_c\tau_d\tau_e\tau_{f]}) = 0 \tag{C.36}$$

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_c\tau_d\tau_e\tau_f\tau_{g]}) = 0 \tag{C.37}$$

$$\mathrm{Tr}(\tau_{[a}\tau_b\tau_c\tau_d\tau_e\tau_f\tau_g\tau_{h]}) = 0 \tag{C.38}$$

Appendix D

$SU(3)$ left/right invariant 1-forms

Left invariant 1-forms

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{1} = &-\Big(\cos c\cos\beta\sin b(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})\\
&\quad+\cos\tfrac{\theta}{2}\sin\beta(\cos b\cos c\cos(a+\gamma)-\sin c\sin(a+\gamma))\Big)d\alpha\\
&+\cos\tfrac{\theta}{2}(\cos(a+\gamma)\sin c+\cos b\cos c\sin(a+\gamma))d\beta\\
&-\cos c\sin b(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})d\gamma-\cos c\sin b\,da+\sin c\,db
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{2} = &\Big(\cos\beta\sin b\sin c(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})\\
&\quad+\cos\tfrac{\theta}{2}\sin\beta(\cos b\cos(a+\gamma)\sin c+\cos c\sin(a+\gamma))\Big)d\alpha\\
&+\cos\tfrac{\theta}{2}(\cos c\cos(a+\gamma)-\cos b\sin c\sin(a+\gamma))d\beta\\
&+\sin b\sin c(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})d\gamma+\sin b\sin c\,da+\cos c\,db
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{3} = &\Big(\cos b\cos\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})-\cos(a+\gamma)\cos\tfrac{\theta}{2}\sin b\sin\beta\Big)d\alpha\\
&+\cos\tfrac{\theta}{2}\sin b\sin(a+\gamma)d\beta+\cos b(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})d\gamma+\cos b\,da+dc
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{4} = &\sin\tfrac{\theta}{2}\Big(\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\beta-\cos\tfrac{b}{2}\cos\beta\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\Big)d\alpha\\
&-\sin\tfrac{b}{2}\sin\tfrac{\theta}{2}\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}d\beta-\tfrac{1}{2}\cos\tfrac{b}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\theta\,d\gamma\\
&+\cos\tfrac{b}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}d\theta
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{5} = &\sin\tfrac{\theta}{2}\Big(\sin\tfrac{b}{2}\sin\beta\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}+\cos\tfrac{b}{2}\cos\beta\cos\tfrac{\theta}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)d\alpha\\
&+\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\tfrac{\theta}{2}d\beta+\tfrac{1}{2}\cos\tfrac{b}{2}\sin\theta\sin\tfrac{a+c+\sqrt{3}\phi}{2}d\gamma\\
&+\cos\tfrac{b}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}d\theta
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{6} = &\sin\tfrac{\theta}{2}\Big(\cos\beta\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}+\cos\tfrac{b}{2}\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\sin\beta\Big)d\alpha\\
&-\cos\tfrac{b}{2}\sin\tfrac{\theta}{2}\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}d\beta+\tfrac{1}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\theta\,d\gamma\\
&-\sin\tfrac{b}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}d\theta
\end{aligned}
$$

$$
\begin{aligned}
\overset{\mathsf{L}}{\omega}{}^{7} = &\sin\tfrac{\theta}{2}\Big(\cos\tfrac{b}{2}\sin\beta\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}-\cos\beta\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\Big)d\alpha\\
&+\cos\tfrac{b}{2}\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{\theta}{2}d\beta-\tfrac{1}{2}\sin\tfrac{b}{2}\sin\theta\sin\tfrac{a-c+\sqrt{3}\phi}{2}d\gamma\\
&-\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}d\theta
\end{aligned}
$$

$$\overset{L}{\omega}{}^8 = -\tfrac{\sqrt{3}}{2}\cos\beta\sin^2\tfrac{\theta}{2}d\alpha - \tfrac{\sqrt{3}}{2}\sin^2\tfrac{\theta}{2}d\gamma + d\phi$$

Right invariant 1-forms

$$\begin{aligned}\overset{R}{\omega}{}^1 = &-\sin\alpha\, d\beta + \cos\alpha\sin\beta\, d\gamma + \cos\alpha\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})da\\ &-\cos\tfrac{\theta}{2}(\cos(a+\gamma)\sin\alpha + \cos\alpha\cos\beta\sin(a+\gamma))db\\ &+\Big(\cos\tfrac{\theta}{2}\sin b(\cos\alpha\cos\beta\cos(a+\gamma) - \sin\alpha\sin(a+\gamma))\\ &\qquad + \cos\alpha\cos b\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})\Big)dc - \tfrac{\sqrt{3}}{2}\cos\alpha\sin\beta\sin^2\tfrac{\theta}{2}d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^2 = &\cos\alpha\, d\beta + \sin\alpha\sin\beta\, d\gamma + \sin\alpha\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})da\\ &+\cos\tfrac{\theta}{2}(\cos\alpha\cos(a+\gamma) - \cos\beta\sin\alpha\sin(a+\gamma))db\\ &+(\cos\tfrac{\theta}{2}\sin b(\cos\beta\cos(a+\gamma)\sin\alpha + \cos\alpha\sin(a+\gamma))\\ &+\cos b\sin\alpha\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2}))dc - \tfrac{\sqrt{3}}{2}\sin\alpha\sin\beta\sin^2\tfrac{\theta}{2}d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^3 = &d\alpha + \cos\beta d\gamma + \cos\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})da + \cos\tfrac{\theta}{2}\sin\beta\sin(a+\gamma)db\\ &+\Big(\cos b\cos\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2}) - \cos(a+\gamma)\cos\tfrac{\theta}{2}\sin b\sin\beta\Big)dc\\ &-\tfrac{\sqrt{3}}{2}\cos\beta\sin^2\tfrac{\theta}{2}d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^4 = &-\cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}d\theta + \tfrac{1}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}\sin\theta da + \sin\tfrac{\beta}{2}\sin\tfrac{2a-\alpha+\gamma}{2}\sin\tfrac{\theta}{2}db\\ &+\sin\tfrac{\theta}{2}\Big(\cos b\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{\alpha+\gamma}{2} - \cos\tfrac{2a-\alpha+\gamma}{2}\sin b\sin\tfrac{\beta}{2}\Big)dc\\ &+\tfrac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}\sin\theta d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^5 = &\cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}d\theta + \tfrac{1}{2}\cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\sin\theta da + \cos\tfrac{2a-\alpha+\gamma}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\theta}{2}db\\ &+\sin\tfrac{\theta}{2}\Big(\sin b\sin\tfrac{\beta}{2}\sin\tfrac{2a-\alpha+\gamma}{2} + \cos b\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\alpha+\gamma}{2}\Big)dc\\ &+\tfrac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\sin\theta d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^6 = &\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}d\theta + \tfrac{1}{2}\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\sin\theta da - \cos\tfrac{\beta}{2}\sin\tfrac{2a+\alpha+\gamma}{2}\sin\tfrac{\theta}{2}db\\ &+\sin\tfrac{\theta}{2}\Big(\cos\tfrac{\beta}{2}\cos\tfrac{2a+\alpha+\gamma}{2}\sin b + \cos b\cos\tfrac{\theta}{2}\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\Big)dc\\ &+\tfrac{\sqrt{3}}{2}\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\sin\theta d\phi\end{aligned}$$

$$\begin{aligned}\overset{R}{\omega}{}^7 = &\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}d\theta - \tfrac{1}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\sin\theta da - \cos\tfrac{\beta}{2}\cos\tfrac{2a+\alpha+\gamma}{2}\sin\tfrac{\theta}{2}db\\ &-\sin\tfrac{\theta}{2}\Big(\cos\tfrac{\beta}{2}\sin b\sin\tfrac{2a+\alpha+\gamma}{2} + \cos b\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\Big)dc\\ &-\tfrac{\sqrt{3}}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\sin\theta d\phi\end{aligned}$$

$$\overset{R}{\omega}{}^8 = -\tfrac{\sqrt{3}}{2}\sin^2\tfrac{\theta}{2}da - \tfrac{\sqrt{3}}{2}\cos b\sin^2\tfrac{\theta}{2}dc + (1-\tfrac{3}{2}\sin^2\tfrac{\theta}{2})d\phi$$

Appendix E

$SU(3)$ left/right invariant vectors

Left invariant vectors

$$\overset{\mathsf{L}}{\xi}_1 = -\frac{\cos c}{\sin b}\,\partial_5 + \sin c\,\partial_6 + \cot b \cos c\,\partial_7$$

$$\overset{\mathsf{L}}{\xi}_2 = \frac{\sin c}{\sin b}\,\partial_5 + \cos c\,\partial_6 - \cot b \sin c\,\partial_7$$

$$\overset{\mathsf{L}}{\xi}_3 = \partial_7$$

$$\begin{aligned}
\overset{\mathsf{L}}{\xi}_4 = {}& \frac{\sin\frac{b}{2}}{\sin\beta\sin\frac{\theta}{2}}\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\partial_1 - \frac{\sin\frac{b}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\partial_2 \\
& -\left(\frac{\sin\frac{b}{2}}{\sin\frac{\theta}{2}}\cot\beta\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2} + \frac{2\cos\frac{b}{2}}{\sin\theta}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\right)\partial_3 \\
& +\cos\tfrac{b}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\partial_4 + \frac{1}{2}\left(\frac{\cot\frac{\theta}{2}}{\cos\frac{b}{2}} + \cos\tfrac{b}{2}\tan\tfrac{\theta}{2}\right)\cos\tfrac{a+c+\sqrt{3}\phi}{2}\partial_5 \\
& -\cot\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\partial_6 + \frac{\cot\frac{\theta}{2}}{2\cos\frac{b}{2}}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\partial_7 \\
& -\frac{\sqrt{3}}{2}\cos\tfrac{b}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\tan\tfrac{\theta}{2}\partial_8
\end{aligned}$$

$$\begin{aligned}
\overset{\mathsf{L}}{\xi}_5 = {}& \frac{\sin\frac{b}{2}}{\sin\beta\sin\frac{\theta}{2}}\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\partial_1 + \frac{\sin\frac{b}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\partial_2 \\
& +\left(\frac{2\cos\frac{b}{2}}{\sin\theta}\sin\tfrac{a+c+\sqrt{3}\phi}{2} - \frac{\sin\frac{b}{2}}{\sin\frac{\theta}{2}}\cot\beta\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\right)\partial_3 \\
& +\cos\tfrac{b}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\partial_4 - \frac{1}{2}\left(\frac{\cot\frac{\theta}{2}}{\cos\frac{b}{2}} + \cos\tfrac{b}{2}\tan\tfrac{\theta}{2}\right)\sin\tfrac{a+c+\sqrt{3}\phi}{2}\partial_5 \\
& -\cos\tfrac{a+c+\sqrt{3}\phi}{2}\cot\tfrac{\theta}{2}\sin\tfrac{b}{2}\partial_6 - \frac{\cot\frac{\theta}{2}}{2\cos\frac{b}{2}}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\partial_7 \\
& +\frac{\sqrt{3}}{2}\cos\tfrac{b}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\tan\tfrac{\theta}{2}\partial_8
\end{aligned}$$

$$\overset{\mathsf{L}}{\xi}_6 = \frac{\cos\frac{b}{2}}{\sin\beta\sin\frac{\theta}{2}}\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\partial_1 - \frac{\cos\frac{b}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\partial_2$$
$$+\left(\frac{2\sin\frac{b}{2}}{\sin\theta}\cos\tfrac{a-c+\sqrt{3}\phi}{2} - \frac{\cos\frac{b}{2}}{\sin\frac{\theta}{2}}\cot\beta\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\right)\partial_3$$
$$-\sin\tfrac{b}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\partial_4 - \frac{1}{2}\left(\frac{\cot\frac{\theta}{2}}{\sin\frac{b}{2}} + \sin\tfrac{b}{2}\tan\tfrac{\theta}{2}\right)\cos\tfrac{a-c+\sqrt{3}\phi}{2}\partial_5$$
$$-\cos\tfrac{b}{2}\cot\tfrac{\theta}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\partial_6 + \frac{\cot\frac{\theta}{2}}{2\sin\frac{b}{2}}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\partial_7$$
$$+\frac{\sqrt{3}}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{L}}{\xi}_7 = \frac{\cos\frac{b}{2}}{\sin\beta\sin\frac{\theta}{2}}\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\partial_1 + \frac{\cos\frac{b}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\partial_2$$
$$-\left(\frac{\cos\frac{b}{2}}{\sin\frac{\theta}{2}}\cot\beta\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2} + \frac{2\sin\frac{b}{2}}{\sin\theta}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\right)\partial_3$$
$$-\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\partial_4 + \frac{1}{2}\left(\frac{\cot\frac{\theta}{2}}{\sin\frac{b}{2}} + \sin\tfrac{b}{2}\tan\tfrac{\theta}{2}\right)\sin\tfrac{a-c+\sqrt{3}\phi}{2}\partial_5$$
$$-\cos\tfrac{b}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\cot\tfrac{\theta}{2}\partial_6 - \frac{\cot\frac{\theta}{2}}{2\sin\frac{b}{2}}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\partial_7$$
$$-\frac{\sqrt{3}}{2}\sin\tfrac{b}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{L}}{\xi}_8 = \partial_8$$

right invariant vectors

$$\overset{\mathsf{R}}{\xi}_1 = -\cos\alpha\cot\beta\,\partial_1 - \sin\alpha\,\partial_2 + \frac{\cos\alpha}{\sin\beta}\partial_3$$

$$\overset{\mathsf{R}}{\xi}_2 = -\sin\alpha\cot\beta\,\partial_1 + \cos\alpha\,\partial_2 + \frac{\sin\alpha}{\sin\beta}\partial_3$$

$$\overset{\mathsf{R}}{\xi}_3 = \partial_1$$

$$\overset{\mathsf{R}}{\xi}_4 = -\frac{\cot\frac{\theta}{2}}{2\cos\frac{\beta}{2}}\cos\tfrac{\alpha+\gamma}{2}\partial_1 + \cot\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\partial_2$$
$$+\cos\tfrac{\alpha+\gamma}{2}\left(\cos\tfrac{\beta}{2}\tan\tfrac{\theta}{2} - \frac{\cot\frac{\theta}{2}}{2\cos\frac{\beta}{2}}\right)\partial_3 - \cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\partial_4$$
$$+\left(\frac{\cot b}{\sin\frac{\theta}{2}}\cos\tfrac{2a-\alpha+\gamma}{2}\sin\tfrac{\beta}{2} + \frac{\cos\frac{\beta}{2}}{\sin\theta}\cos\tfrac{\alpha+\gamma}{2}(2-3\sin^2\tfrac{\theta}{2})\right)\partial_5$$
$$+\frac{\sin\frac{\beta}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{2a-\alpha+\gamma}{2}\partial_6 - \frac{\sin\frac{\beta}{2}}{\sin b\sin\frac{\theta}{2}}\cos\tfrac{2a-\alpha+\gamma}{2}\partial_7$$
$$+\frac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{R}}{\xi}_5 = -\frac{\cot\frac{\theta}{2}}{2\cos\frac{\beta}{2}}\sin\tfrac{\alpha+\gamma}{2}\partial_1 - \cos\tfrac{\alpha+\gamma}{2}\cot\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\partial_2$$
$$+\sin\tfrac{\alpha+\gamma}{2}\left(\cos\tfrac{\beta}{2}\tan\tfrac{\theta}{2} - \frac{\cot\frac{\theta}{2}}{2\cos\frac{\beta}{2}}\right)\partial_3 + \cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}\partial_4$$
$$+\left(\frac{\cos\frac{\beta}{2}}{\sin\theta}\sin\tfrac{\alpha+\gamma}{2}(2-3\sin^2\tfrac{\theta}{2}) - \frac{\cot b}{\sin\frac{\theta}{2}}\sin\tfrac{2a-\alpha+\gamma}{2}\sin\tfrac{\beta}{2}\right)\partial_5$$
$$+\frac{\sin\frac{\beta}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{2a-\alpha+\gamma}{2}\partial_6 + \frac{\sin\frac{\beta}{2}}{\sin b\sin\frac{\theta}{2}}\sin\tfrac{2a-\alpha+\gamma}{2}\partial_7$$
$$+\frac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{R}}{\xi}_6 = \frac{\cot\frac{\theta}{2}}{2\sin\frac{\beta}{2}}\cos\tfrac{\alpha-\gamma}{2}\partial_1 + \cos\tfrac{\beta}{2}\cot\tfrac{\theta}{2}\sin\tfrac{\alpha-\gamma}{2}\partial_2$$
$$+\cos\tfrac{\alpha-\gamma}{2}\left(\sin\tfrac{\beta}{2}\tan\tfrac{\theta}{2} - \frac{\cot\frac{\theta}{2}}{2\sin\frac{\beta}{2}}\right)\partial_3 + \sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\partial_4$$
$$+\left(\frac{\sin\frac{\beta}{2}}{\sin\theta}\cos\tfrac{\alpha-\gamma}{2}(2-3\sin^2\tfrac{\theta}{2}) - \frac{\cot b}{\sin\frac{\theta}{2}}\cos\tfrac{2a+\alpha+\gamma}{2}\cos\tfrac{\beta}{2}\right)\partial_5$$
$$-\frac{\cos\frac{\beta}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{2a+\alpha+\gamma}{2}\partial_6 + \frac{\cos\frac{\beta}{2}}{\sin b\sin\frac{\theta}{2}}\cos\tfrac{2a+\alpha+\gamma}{2}\partial_7$$
$$+\frac{\sqrt{3}}{2}\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{R}}{\xi}_7 = -\frac{\cot\frac{\theta}{2}}{2\sin\frac{\beta}{2}}\sin\tfrac{\alpha-\gamma}{2}\partial_1 + \cos\tfrac{\beta}{2}\cos\tfrac{\alpha-\gamma}{2}\cot\tfrac{\theta}{2}\partial_2$$
$$+\sin\tfrac{\alpha-\gamma}{2}\left(\frac{\cot\frac{\theta}{2}}{2\sin\frac{\beta}{2}} - \sin\tfrac{\beta}{2}\tan\tfrac{\theta}{2}\right)\partial_3 + \cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\partial_4$$
$$+\left(\frac{\cot b}{\sin\frac{\theta}{2}}\cos\tfrac{\beta}{2}\sin\tfrac{2a+\alpha+\gamma}{2} - \frac{\sin\frac{\beta}{2}}{\sin\theta}\sin\tfrac{\alpha-\gamma}{2}(2-3\sin^2\tfrac{\theta}{2})\right)\partial_5$$
$$-\frac{\cos\frac{\beta}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{2a+\alpha+\gamma}{2}\partial_6 - \frac{\cos\frac{\beta}{2}}{\sin b\sin\frac{\theta}{2}}\sin\tfrac{2a+\alpha+\gamma}{2}\partial_7$$
$$-\frac{\sqrt{3}}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\tan\tfrac{\theta}{2}\partial_8$$

$$\overset{\mathsf{R}}{\xi}_8 = \sqrt{3}\partial_3 - \sqrt{3}\partial_5 + \partial_8$$

Appendix F

The adjoint matrix for $SU(3)$ invariant forms and vectrors

$$
\begin{aligned}
r[1,1] = & -\cos c\cos\alpha\sin b\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2}) \\
& -\cos\tfrac{\theta}{2}\sin c(\cos(a+\gamma)\sin\alpha+\cos\alpha\cos\beta\sin(a+\gamma)) \\
& +\cos b\cos c\cos\tfrac{\theta}{2}(\cos\alpha\cos\beta\cos(a+\gamma)-\sin\alpha\sin(a+\gamma))
\end{aligned}
$$

$$
\begin{aligned}
r[1,2] = & \cos\alpha\cos\tfrac{\theta}{2}(\cos(a+\gamma)\sin c+\cos b\cos c\sin(a+\gamma)) \\
& -\sin\alpha\Big(\cos\beta\cos\tfrac{\theta}{2}\sin c\sin(a+\gamma) \\
& \quad +\cos c(\sin b\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})-\cos b\cos\beta\cos(a+\gamma)\cos\tfrac{\theta}{2})\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[1,3] = & -\cos c\Big(\cos\beta\sin b(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})+\cos b\cos(a+\gamma)\cos\tfrac{\theta}{2}\sin\beta\Big) \\
& +\cos\tfrac{\theta}{2}\sin c\sin\beta\sin(a+\gamma)
\end{aligned}
$$

$$
\begin{aligned}
r[1,4] = & \sin\tfrac{\theta}{2}\Big(\sin\tfrac{\beta}{2}(-\cos b\cos c\cos\tfrac{2a-\alpha+\gamma}{2}+\sin c\sin\tfrac{2a-\alpha+\gamma}{2}) \\
& \quad -\cos c\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{\alpha+\gamma}{2}\sin b\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[1,5] = & \sin\tfrac{\theta}{2}\Big(\sin\tfrac{\beta}{2}(\cos\tfrac{2a-\alpha+\gamma}{2}\sin c+\cos b\cos c\sin\tfrac{2a-\alpha+\gamma}{2}) \\
& -\cos c\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin b\sin\tfrac{\alpha+\gamma}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[1,6] = & \sin\tfrac{\theta}{2}\Big(\cos\tfrac{\beta}{2}(\cos b\cos c\cos\tfrac{2a+\alpha+\gamma}{2}-\sin c\sin\tfrac{2a+\alpha+\gamma}{2}) \\
& \quad -\cos c\cos\tfrac{\alpha-\gamma}{2}\cos\tfrac{\theta}{2}\sin b\sin\tfrac{\beta}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[1,7] = & \sin\tfrac{\theta}{2}\Big(-\cos\tfrac{\beta}{2}(\cos\tfrac{2a+\alpha+\gamma}{2}\sin c+\cos b\cos c\sin\tfrac{2a+\alpha+\gamma}{2}) \\
& +\cos c\cos\tfrac{\theta}{2}\sin b\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\Big)
\end{aligned}
$$

$$
r[1,8] = \tfrac{\sqrt{3}}{2}\cos c\sin b\sin^2\tfrac{\theta}{2}
$$

$$
\begin{aligned}
r[2,1] = {} & \cos\alpha\sin b\sin c\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2}) \\
& -\cos c\cos\tfrac{\theta}{2}(\cos(a+\gamma)\sin\alpha+\cos\alpha\cos\beta\sin(a+\gamma)) \\
& +\cos b\cos\tfrac{\theta}{2}\sin c(\sin\alpha\sin(a+\gamma)-\cos\alpha\cos\beta\cos(a+\gamma))
\end{aligned}
$$

$$
\begin{aligned}
r[2,2] = {} & \cos c\cos\tfrac{\theta}{2}(\cos\alpha\cos(a+\gamma)-\cos\beta\sin\alpha\sin(a+\gamma)) \\
& +\sin c\Big(\sin b\sin\alpha\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2}) \\
& \quad -\cos b\cos\tfrac{\theta}{2}(\cos\beta\cos(a+\gamma)\sin\alpha+\cos\alpha\sin(a+\gamma))\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[2,3] = {} & \cos\tfrac{\theta}{2}\sin\beta(\cos b\cos(a+\gamma)\sin c+\cos c\sin(a+\gamma)) \\
& +\cos\beta\sin b\sin c(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})
\end{aligned}
$$

$$
\begin{aligned}
r[2,4] = {} & \sin\tfrac{\theta}{2}\Big(\sin\tfrac{\beta}{2}(\cos b\cos\tfrac{2a-\alpha+\gamma}{2}\sin c+\cos c\sin\tfrac{2a-\alpha+\gamma}{2}) \\
& +\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{\alpha+\gamma}{2}\sin b\sin c\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[2,5] = {} & \sin\tfrac{\theta}{2}\Big(\sin c(-\cos b\sin\tfrac{\beta}{2}\sin\tfrac{2a-\alpha+\gamma}{2}+\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin b\sin\tfrac{\alpha+\gamma}{2}) \\
& \quad +\cos c\cos\tfrac{2a-\alpha+\gamma}{2}\sin\tfrac{\beta}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[2,6] = {} & \sin\tfrac{\theta}{2}\Big(-\cos\tfrac{\beta}{2}(\cos b\cos\tfrac{2a+\alpha+\gamma}{2}\sin c+\cos c\sin\tfrac{2a+\alpha+\gamma}{2}) \\
& \quad +\cos\tfrac{\theta}{2}\cos\tfrac{\alpha-\gamma}{2}\sin b\sin c\sin\tfrac{\beta}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
r[2,7] = {} & \sin\tfrac{\theta}{2}\Big(\sin c(\cos b\cos\tfrac{\beta}{2}\sin\tfrac{2a+\alpha+\gamma}{2}-\cos\tfrac{\theta}{2}\sin b\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}) \\
& -\cos c\cos\tfrac{\beta}{2}\cos\tfrac{2a+\alpha+\gamma}{2}\Big)
\end{aligned}
$$

$$
r[2,8] = -\tfrac{\sqrt{3}}{2}\sin b\sin c\sin^2\tfrac{\theta}{2}
$$

$$
\begin{aligned}
r[3,1] = {} & \cos\alpha(\cos\beta\cos(a+\gamma)\cos\tfrac{\theta}{2}\sin b+\cos b\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})) \\
& -\cos\tfrac{\theta}{2}\sin b\sin\alpha\sin(a+\gamma)
\end{aligned}
$$

$$
\begin{aligned}
r[3,2] = {} & \cos\tfrac{\theta}{2}\sin b(\cos\beta\cos(a+\gamma)\sin\alpha+\cos\alpha\sin(a+\gamma)) \\
& +\cos b\sin\alpha\sin\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})
\end{aligned}
$$

$$
r[3,3] = \cos b\cos\beta(1-\tfrac{1}{2}\sin^2\tfrac{\theta}{2})-\cos(a+\gamma)\cos\tfrac{\theta}{2}\sin b\sin\beta
$$

$$
r[3,4] = \sin\tfrac{\theta}{2}(\cos b\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{\alpha+\gamma}{2}-\cos\tfrac{2a-\alpha+\gamma}{2}\sin b\sin\tfrac{\beta}{2})
$$

$$
r[3,5] = \sin\tfrac{\theta}{2}(\sin b\sin\tfrac{\beta}{2}\sin\tfrac{2a-\alpha+\gamma}{2}+\cos b\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\alpha+\gamma}{2})
$$

$$
r[3,6] = \sin\tfrac{\theta}{2}(\cos\tfrac{\beta}{2}\cos\tfrac{2a+\alpha+\gamma}{2}\sin b+\cos b\cos\tfrac{\alpha-\gamma}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2})
$$

$$
r[3,7] = -\sin\tfrac{\theta}{2}(\cos\tfrac{\beta}{2}\sin b\sin\tfrac{2a+\alpha+\gamma}{2}+\cos b\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2})
$$

$$
r[3,8] = -\tfrac{\sqrt{3}}{2}\cos b\sin^2\tfrac{\theta}{2}
$$

$$r[4,1] = \sin\tfrac{b}{2}\sin\alpha\sin\tfrac{\theta}{2}\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2} - \cos\alpha\sin\tfrac{\theta}{2}\Big(\cos\beta\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2} + \cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\beta\Big)$$

$$r[4,2] = -\cos\alpha\sin\tfrac{b}{2}\sin\tfrac{\theta}{2}\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2} - \sin\alpha\sin\tfrac{\theta}{2}\Big(\cos\beta\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2} + \cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\beta\Big)$$

$$r[4,3] = \sin\tfrac{\theta}{2}(\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\beta - \cos\tfrac{b}{2}\cos\beta\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2})$$

$$r[4,4] = -\cos\tfrac{\theta}{2}\cos\tfrac{a-c-\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\tfrac{\beta}{2} + \cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\Big(\cos\tfrac{\alpha+\gamma}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2} - \sin\tfrac{\alpha+\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[4,5] = \cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{\beta}{2}\sin\tfrac{a-c-\alpha+\gamma-\sqrt{3}\phi}{2} + \cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\Big(\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha+\gamma}{2} + \cos\tfrac{\alpha+\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[4,6] = \cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2} + \cos\tfrac{b}{2}\sin\tfrac{\beta}{2}\Big(\cos\tfrac{\alpha-\gamma}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2} + \sin\tfrac{\alpha-\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[4,7] = -\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{a-c+\alpha+\gamma-\sqrt{3}\phi}{2} + \cos\tfrac{b}{2}\sin\tfrac{\beta}{2}\Big(-\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha-\gamma}{2} + \cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[4,8] = -\tfrac{\sqrt{3}}{2}\cos\tfrac{b}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\theta$$

$$r[5,1] = \sin\tfrac{\theta}{2}\Big(\cos\tfrac{b}{2}\cos\alpha\cos\tfrac{\theta}{2}\sin\beta\sin\tfrac{a+c+\sqrt{3}\phi}{2} - \sin\tfrac{b}{2}(\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2}\sin\alpha + \cos\alpha\cos\beta\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2})\Big)$$

$$r[5,2] = \sin\tfrac{\theta}{2}\Big(\cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\sin\alpha\sin\beta\sin\tfrac{a+c+\sqrt{3}\phi}{2} + \sin\tfrac{b}{2}(\cos\alpha\cos\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2} - \cos\beta\sin\alpha\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2})\Big)$$

$$r[5,3] = \sin\tfrac{\theta}{2}(\sin\tfrac{b}{2}\sin\beta\sin\tfrac{a-c+2\gamma-\sqrt{3}\phi}{2} + \cos\tfrac{b}{2}\cos\beta\cos\tfrac{\theta}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2})$$

$$r[5,4] = -\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{\beta}{2}\sin\tfrac{a-c-\alpha+\gamma-\sqrt{3}\phi}{2} - \cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\Big(\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha+\gamma}{2} + \cos\tfrac{\alpha+\gamma}{2}\cos\theta\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[5,5] = -\cos\tfrac{\theta}{2}\cos\tfrac{a-c-\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\tfrac{\beta}{2} + \cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\Big(\cos\tfrac{\alpha+\gamma}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2} - \cos\theta\sin\tfrac{\alpha+\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[5,6] = \cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{a-c+\alpha+\gamma-\sqrt{3}\phi}{2} + \cos\tfrac{b}{2}\sin\tfrac{\beta}{2}\Big(\cos\tfrac{a+c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha-\gamma}{2} - \cos\tfrac{\alpha-\gamma}{2}\cos\theta\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[5,7] = \cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{b}{2} + \cos\tfrac{b}{2}\sin\tfrac{\beta}{2}\Big(\cos\tfrac{\alpha-\gamma}{2}\cos\tfrac{a+c+\sqrt{3}\phi}{2} + \cos\theta\sin\tfrac{\alpha-\gamma}{2}\sin\tfrac{a+c+\sqrt{3}\phi}{2}\Big)$$

$$r[5,8] = \tfrac{\sqrt{3}}{2}\cos\tfrac{b}{2}\sin\theta\sin\tfrac{a+c+\sqrt{3}\phi}{2}$$

$$\begin{aligned} r[6,1] = \sin\tfrac{\theta}{2}\big(&\cos\alpha\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\beta \\ &+\cos\tfrac{b}{2}(\sin\alpha\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2} - \cos\alpha\cos\beta\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2})\big) \end{aligned}$$

$$\begin{aligned} r[6,2] = \sin\tfrac{\theta}{2}\big(&\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\alpha\sin\beta \\ &-\cos\tfrac{b}{2}(\cos\beta\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\sin\alpha + \cos\alpha\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2})\big) \end{aligned}$$

$$r[6,3] = \sin\tfrac{\theta}{2}(\cos\beta\cos\tfrac{\theta}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2} + \cos\tfrac{b}{2}\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\sin\beta)$$

$$\begin{aligned} r[6,4] = &-\cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c-\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{\beta}{2} + \cos\tfrac{\beta}{2}\sin\tfrac{b}{2} \\ &(\sin\tfrac{\alpha+\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2} - \cos\tfrac{\alpha+\gamma}{2}\cos\theta\cos\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$\begin{aligned} r[6,5] = &\cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\tfrac{a+c-\alpha+\gamma-\sqrt{3}\phi}{2} - \cos\tfrac{\beta}{2}\sin\tfrac{b}{2} \\ &(\cos\theta\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha+\gamma}{2} + \cos\tfrac{\alpha+\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$\begin{aligned} r[6,6] = &\cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\alpha+\gamma-\sqrt{3}\phi}{2} - \sin\tfrac{b}{2}\sin\tfrac{\beta}{2} \\ &(\cos\tfrac{\alpha-\gamma}{2}\cos\theta\cos\tfrac{a-c+\sqrt{3}\phi}{2} + \sin\tfrac{\alpha-\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$\begin{aligned} r[6,7] = &-\cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{a+c+\alpha+\gamma-\sqrt{3}\phi}{2} + \sin\tfrac{b}{2}\sin\tfrac{\beta}{2} \\ &(\cos\theta\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha-\gamma}{2} - \cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$r[6,8] = \tfrac{\sqrt{3}}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{b}{2}\sin\theta$$

$$\begin{aligned} r[7,1] = -\sin\tfrac{\theta}{2}\big(&\cos\alpha\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\beta\sin\tfrac{a-c+\sqrt{3}\phi}{2} \\ &+\cos\tfrac{b}{2}(\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2}\sin\alpha + \cos\alpha\cos\beta\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2})\big) \end{aligned}$$

$$\begin{aligned} r[7,2] = \sin\tfrac{\theta}{2}\big(&-\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\alpha\sin\beta\sin\tfrac{a-c+\sqrt{3}\phi}{2} \\ &+\cos\tfrac{b}{2}(\cos\alpha\cos\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2} - \cos\beta\sin\alpha\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2})\big) \end{aligned}$$

$$r[7,3] = \sin\tfrac{\theta}{2}(\cos\beta\cos\tfrac{\theta}{2}\sin\tfrac{b}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2} - \cos\tfrac{b}{2}\sin\beta\sin\tfrac{a+c+2\gamma-\sqrt{3}\phi}{2})$$

$$\begin{aligned} r[7,4] = &-\cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\tfrac{a+c-\alpha+\gamma-\sqrt{3}\phi}{2} + \cos\tfrac{\beta}{2}\sin\tfrac{b}{2} \\ &(\cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha+\gamma}{2} + \cos\tfrac{\alpha+\gamma}{2}\cos\theta\sin\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$\begin{aligned} r[7,5] = &-\cos\tfrac{b}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c-\alpha+\gamma-\sqrt{3}\phi}{2}\sin\tfrac{\beta}{2} + \cos\tfrac{\beta}{2}\sin\tfrac{b}{2} \\ &(\cos\theta\sin\tfrac{\alpha+\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2} - \cos\tfrac{\alpha+\gamma}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2}) \end{aligned}$$

$$\begin{aligned} r[7,6] = &\cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{a+c+\alpha+\gamma-\sqrt{3}\phi}{2} + \sin\tfrac{b}{2}\sin\tfrac{\beta}{2} \\ &(\cos\tfrac{\alpha-\gamma}{2}\cos\theta\sin\tfrac{a-c+\sqrt{3}\phi}{2} - \cos\tfrac{a-c+\sqrt{3}\phi}{2}\sin\tfrac{\alpha-\gamma}{2}) \end{aligned}$$

$$r[7,7] = \cos\tfrac{b}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\theta}{2}\cos\tfrac{a+c+\alpha+\gamma-\sqrt{3}\phi}{2} - \sin\tfrac{b}{2}\sin\tfrac{\beta}{2}$$
$$(\cos\tfrac{\alpha-\gamma}{2}\cos\tfrac{a-c+\sqrt{3}\phi}{2} + \cos\theta\sin\tfrac{\alpha-\gamma}{2}\sin\tfrac{a-c+\sqrt{3}\phi}{2})$$

$$r[7,8] = -\tfrac{\sqrt{3}}{2}\sin\tfrac{b}{2}\sin\theta\sin\tfrac{a-c+\sqrt{3}\phi}{2}$$

$$r[8,1] = -\tfrac{\sqrt{3}}{2}\cos\alpha\sin\beta\sin^2\tfrac{\theta}{2}$$

$$r[8,2] = -\tfrac{\sqrt{3}}{2}\sin\alpha\sin\beta\sin^2\tfrac{\theta}{2}$$

$$r[8,3] = -\tfrac{\sqrt{3}}{2}\cos\beta\sin^2\tfrac{\theta}{2}$$

$$r[8,4] = \tfrac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\alpha+\gamma}{2}\sin\theta$$

$$r[8,5] = \tfrac{\sqrt{3}}{2}\cos\tfrac{\beta}{2}\sin\tfrac{\alpha+\gamma}{2}\sin\theta$$

$$r[8,6] = \tfrac{\sqrt{3}}{2}\cos\tfrac{\alpha-\gamma}{2}\sin\tfrac{\beta}{2}\sin\theta$$

$$r[8,7] = -\tfrac{\sqrt{3}}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\alpha-\gamma}{2}\sin\theta$$

$$r[8,8] = 1 - \tfrac{3}{2}\sin^2\tfrac{\theta}{2}$$

Appendix G

$SU(3)$ harmonic forms

3-form

N	$\mu\nu\lambda$	$\psi_{\mu\nu\lambda}$
1	123	$-\sin\beta$
2	125	$-(1-\frac{1}{2}\sin^2\frac{\theta}{2})\sin\beta$
3	126	$\cos\beta\cos\frac{\theta}{2}\sin(a+\gamma)$
4	127	$-\cos\beta\cos(a+\gamma)\cos\frac{\theta}{2}\sin b-\cos b\sin\beta(1-\frac{1}{2}\sin^2\frac{\theta}{2})$
5	128	$\frac{\sqrt{3}}{2}\sin\beta\sin^2\frac{\theta}{2}$
6	136	$\cos(a+\gamma)\cos\frac{\theta}{2}\sin\beta$
7	137	$\cos\frac{\theta}{2}\sin b\sin\beta\sin(a+\gamma)$
8	145	$-\frac{1}{4}\cos\beta\sin\theta$
9	146	$-\frac{1}{2}\sin\beta\sin(a+\gamma)\sin\frac{\theta}{2}$
10	147	$\frac{1}{2}\left(-\cos b\cos\beta\cos\frac{\theta}{2}+\cos(a+\gamma)\sin b\sin\beta\right)\sin\frac{\theta}{2}$
11	148	$-\frac{\sqrt{3}}{4}\cos\beta\sin\theta$
12	156	$\cos(a+\gamma)\cos\frac{\theta}{2}\sin\beta$
13	157	$\cos\frac{\theta}{2}\sin b\sin\beta\sin(a+\gamma)$
14	167	$-\cos\beta(1-\frac{1}{2}\sin^2\frac{\theta}{2})\sin b-\cos b\cos(a+\gamma)\cos\frac{\theta}{2}\sin\beta$
15	236	$-\cos\frac{\theta}{2}\sin(a+\gamma)$
16	237	$\cos(a+\gamma)\cos\frac{\theta}{2}\sin b$
17	246	$-\frac{1}{2}\cos(a+\gamma)\sin\frac{\theta}{2}$
18	247	$-\frac{1}{2}\sin b\sin(a+\gamma)\sin\frac{\theta}{2}$
19	256	$-\cos\frac{\theta}{2}\sin(a+\gamma)$
20	257	$\cos(a+\gamma)\cos\frac{\theta}{2}\sin b$

21	267	$\cos b \cos \frac{\theta}{2} \sin(a+\gamma)$
22	345	$-\frac{1}{4} \sin\theta$
23	347	$-\frac{1}{4} \cos b \sin\theta$
24	348	$-\frac{\sqrt{3}}{4} \sin\theta$
25	367	$-(1-\frac{1}{2}\sin^2\frac{\theta}{2}) \sin b$
26	567	$-\sin b$

5-form

N	$\mu\nu\lambda\rho\sigma$	$\Upsilon_{\mu\nu\lambda\rho\sigma}$
1	12345	$\frac{\sqrt{3}}{2} \sin\beta \sin\theta$
2	12347	$\frac{\sqrt{3}}{2} \cos b \sin\beta \sin\theta$
3	12348	$\frac{3}{2} \sin\beta \sin\theta$
4	12367	$-\sqrt{3} \sin b \sin\beta \sin^2\frac{\theta}{2}$
5	12456	$-\frac{\sqrt{3}}{8} K \cos\beta \sin(a+\gamma)$
6	12457	$\frac{\sqrt{3}}{8} K \cos\beta \cos(a+\gamma) \sin b$
7	12458	$-\sin\beta \sin\theta$
8	12467	$\frac{\sqrt{3}}{8} K \cos b \cos\beta \sin(a+\gamma)$
9	12468	$\frac{1}{4} \cos\beta(5+3\cos\theta) \sin(a+\gamma) \sin\frac{\theta}{2}$
10	12478	$-\cos b \sin\beta \sin\theta - \frac{1}{8} \cos\beta \cos(a+\gamma) \sin b(7\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2})$
11	12567	$-\sqrt{3} \sin b \sin\beta \sin^2\frac{\theta}{2}$
12	12568	$\cos\beta \cos(a+\gamma) \sin^2\frac{\theta}{2}$
13	12578	$\cos\beta \cos\frac{\theta}{2} \sin b \sin(a+\gamma) \sin^2\frac{\theta}{2}$
14	12678	$(\sin b \sin\beta - \cos b \cos\beta \cos(a+\gamma) \cos\frac{\theta}{2}) \sin^2\frac{\theta}{2}$
15	13456	$-\frac{\sqrt{3}}{8} K \cos(a+\gamma) \sin\beta$
16	13457	$-\frac{\sqrt{3}}{8} K \sin b \sin\beta \sin(a+\gamma)$
17	13467	$\frac{\sqrt{3}}{8} K \cos b \cos(a+\gamma) \sin\beta$
18	13468	$\frac{1}{4} \cos(a+\gamma)(5+3\cos\theta) \sin\beta \sin\frac{\theta}{2}$
19	13478	$\frac{1}{4}(5+3\cos\theta) \sin b \sin\beta \sin(a+\gamma) \sin\frac{\theta}{2}$
20	13568	$-\cos\frac{\theta}{2} \sin\beta \sin(a+\gamma) \sin^2\frac{\theta}{2}$
21	13578	$\cos(a+\gamma) \cos\frac{\theta}{2} \sin b \sin\beta \sin^2\frac{\theta}{2}$

22	13678	$\cos b \cos \frac{\theta}{2} \sin \beta \sin(a+\gamma) \sin^2 \frac{\theta}{2}$
23	14567	$\frac{\sqrt{3}}{2} \cos \beta \sin b \sin \theta$
24	14568	$-\cos(a+\gamma) \sin \beta \sin \frac{\theta}{2}$
25	14578	$-\sin b \sin \beta \sin(a+\gamma) \sin \frac{\theta}{2}$
26	14678	$3(\cos \beta \cos \frac{\theta}{2} \sin b + \cos b \cos(a+\gamma) \sin \beta) \sin \frac{\theta}{2}$
27	23456	$\frac{\sqrt{3}}{8} K \sin(a+\gamma)$
28	23457	$-\frac{\sqrt{3}}{8} K \cos(a+\gamma) \sin b$
29	23467	$-\frac{\sqrt{3}}{8} K \cos b \sin(a+\gamma)$
30	23468	$-\frac{1}{4}(5+3\cos\theta) \sin(a+\gamma) \sin \frac{\theta}{2}$
31	23478	$\frac{1}{4} \cos(a+\gamma)(5+3\cos\theta) \sin b \sin \frac{\theta}{2}$
32	23568	$-\cos(a+\gamma) \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2}$
33	23578	$-\cos \frac{\theta}{2} \sin b \sin(a+\gamma) \sin^2 \frac{\theta}{2}$
34	23678	$\cos b \cos(a+\gamma) \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2}$
35	24568	$\sin(a+\gamma) \sin \frac{\theta}{2}$
36	24578	$-\cos(a+\gamma) \sin b \sin \frac{\theta}{2}$
37	24678	$-\cos b \sin(a+\gamma) \sin \frac{\theta}{2}$
38	34567	$\frac{\sqrt{3}}{2} \sin b \sin \theta$
39	34678	$\frac{1}{2} \sin b \sin \theta$

where

$$K = 5 \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \tag{G.1}$$

Bibliography

[1] Y.Choquet-Bruhat, C.DeWitt-Morette and M.Dillard-Bleick, *Analysis, Manifolds and Physics*, revised edition 1982, North-Holland Publishing Company, Amsterdam.

[2] M.A.B.Bég and H.Ruegg, JMP **6** (1965) 677-682.

[3] L.J.Boya, Rep.Math.Phys. **30** (1991) 149-162.

[4] M.Byrd, *Differential geometry on SU(3) with applications to three state systems*, JMP **39** (1998) pp. 6125-6136.

[5] M.Byrd, *Erratum: "Differential geometry on SU(3) with applications to three state systems" [JMP 39 (1998) pp. 6125-6136]*, JMP **41** (2000) pp. 1026-1030.

[6] M.Byrd and E.C.G.Sudarshan, *SU(3) Revisited*, arxiv: physics/9803029.

[7] T.Eguchi, P.B.Gilkey and A.J.Hanson, *"Gravitation, gauge theories and differential geometry"*, Phys.Rep. **66**, N 6 (1980), pp. 213-393.

[8] S.Helgason, *Differential geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics; vol. 34 (Providence, R.I.: AMS, 2001)

[9] G.Khanna, S.Mukhopadhyay, R.Simon and N.Mukunda, Ann.Phys. N.Y. **253** (1997) 55-82.

[10] T.Tilma, M.Byrd and E.C.G.Sudarshan, J.Phys. A: Math, Gen. **35** (2002) 10445-10465.

[11] T.Tilma and E.C.G.Sudarshan, J.Phys. A: Math, Gen. **35** (2002) 10467-10501.

[12] V. Gerdt, R. Horan, A. Khvedelidze, M. Lavelle, D. McMullan and Yu. Palii, *On the Hamiltonian reduction of geodesic motion on SU(3) to SU(3)/SU(2)*, arXiv: hep-th/0511245, submitted to ≪ Journal of Mathematical Physics ≫.