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LIGHT-CONE YANG–MILLS MECHANICS: $SU(2)$ VS. $SU(3)$

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We investigate the light-cone $SU(n)$ Yang–Mills mechanics formulated as the leading order of the long-wavelength approximation to the light-front $SU(n)$ Yang–Mills theory. In the framework of the Dirac formalism for degenerate Hamiltonian systems, for models with the structure groups $SU(2)$ and $SU(3)$, we determine the complete set of constraints and classify them. We show that the light-cone mechanics has an extended invariance: in addition to the local $SU(n)$ gauge rotations, there is a new local two-parameter Abelian transformation, not related to the isotopic group, that leaves the Lagrangian system unchanged. This extended invariance has one profound consequence. It turns out that the light-cone $SU(2)$ Yang–Mills mechanics, in contrast to the well-known instant-time $SU(2)$ Yang–Mills mechanics, represents a classically integrable system. For calculations, we use the technique of Gröbner bases in the theory of polynomial ideals.

Keywords: gauge symmetry, Hamiltonian system, Gröbner basis

1. Introduction

In his conceptual 1949 paper, where he stated a principally new concept of the three forms of relativistic dynamics, Dirac wrote [1] (also see [2]): "There is no conclusive arguments in favor of one or other of the forms. Even if it could be decided that one of them is the most convenient, this would not necessarily be the one chosen by nature, in the event that only one of them is possible for atomic systems. Thus all three forms should be studied further." After this seminal paper, investigations of the field theories showed prinicpal differences between the introduced forms of dynamics. The simplest example of a free scalar field theory formulated in the light-front form already clearly shows this difference. Here, in contrast to the instant-time evolution, the theory is degenerate because of the choice of the light-like variable x^+ as the time parameter. The corresponding *zero modes* of the field Fourier decomposition with respect to the coordinate x^- (see [3] play the principal role in the consistent description of interactions in the formulation of the light-front field theory. Here, we intend to discuss one aspect of the zero-mode dynamics related to the local symmetry manifestation in gauge theories. We consider a mechanical model, the *light-cone $SU(n)$ Yang–Mills gauge mechanics*, that describes the dynamics of the zero modes of the light-cone $SU(n)$ gluodynamics completely separated from all other normal modes. In other words, we study the lowest order of the long-wavelength approximation to the light-front form of the Yang–Mills field theory. It turns out that in comparison with the well-known long-wavelength approximation in the instant form,[1] the light-cone $SU(n)$ Yang–Mills mechanics has several new, unusual features. Among them, we note the dependence of

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[1]The instant $SU(2)$ Yang–Mills mechanics was formulated almost thirty years ago and still is intensively studied as an example of a model exhibiting various interesting dynamical features at both the classical and quantum levels (see, e.g., some issues in [4]–[15] and the references therein).

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the structure and constraint set on the rank of the isotopic group and the presence of additional first-class constraints generated by the new local symmetry of the model. Moreover, it turns that as a result of resolving all constraints, the light-cone $SU(2)$ Yang–Mills mechanics is a classically integrable system. A brief discussion of these issues is the subject of our further exposition.

In Sec. 2, we formulate the light-cone $SU(n)$ Yang–Mills mechanics and present the complete set of constraints in the cases of the structure groups $SU(2)$ and $SU(3)$. In Sec. 3, we construct the generator of gauge transformations taking the new local symmetry of the light-cone mechanics into account. In Sec. 4, we state the results of the Hamiltonian reduction of the $SU(2)$ model. Because the calculations become extremely intricate and tedious for the structure group $SU(3)$, we calculate using a computer algebra program. For this, a calculation method was developed and applied based on an algorithmic approach to commutative algebra, the technique of Gröbner bases [16]–[23]. In the appendix, we present the homogeneous Gröbner basis especially constructed for this purpose.

2. Light-cone Yang–Mills mechanics

Below, we formulate the light-cone $SU(n)$ Yang–Mills mechanics as the model determined by the leading order of the long-wavelength approximation to the light-front $SU(n)$ Yang–Mills field theory. The coordinate-free representation of the action of $SU(n)$ Yang–Mills fields in the four-dimensional Minkowski space M_4 endowed with the metric η is

$$S := \frac{1}{g_0^2}\int_{M_4} \operatorname{tr} F\wedge *F, \tag{1}$$

where g_0 is a coupling constant and the curvature 2-form $F := dA + A\wedge A$ is constructed from the connection 1-form A. The connection and curvature take values in the Lie algebra $su(n)$ in which some basis T^a is chosen,

$$A = A^a T^a, \qquad F = F^a T^a, \quad a = 1, 2, \ldots, n^2-1.$$

The metric η enters the action through the dual field-strength tensor

$$*F_{\mu\nu} := \frac{1}{2}\sqrt{-\det\eta}\;\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

with the totally antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$.

To formulate the light-cone $SU(n)$ mechanics, we write the connection 1-form A in the so-called light-front basis,

$$A := A_+\,dx^+ + A_-\,dx^- + A_k\,dx^k, \quad k = 1, 2, \tag{2}$$

where the basic 1-forms $dx^\pm$ are dual to the vectors $e_\pm := (e_0 \pm e_3)/\sqrt{2}$ tangent to the light-cone. The corresponding light-cone coordinates $x^\mu = (x^+, x^-, x^\perp)$ are

$$x^\pm := \frac{1}{\sqrt{2}}(x^0 \pm x^3), \qquad x^\perp := x^k, \quad k = 1, 2,$$

and the nonzero components of the metric are $\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1$.

Supposing that the components of the connection 1-form A in (2) are functions only of the time coordinate x^+,

$$A_\pm = A_\pm(x^+), \qquad A_k = A_k(x^+),$$

and factoring the spatial volume $V^{(3)}$, we reduce action (1) to

$$S_{\rm lc} := \frac{V^{(3)}}{2g_0^2}\int dx^+\,(F^a_{+-}F^a_{+-} + 2F^a_{+k}F^a_{-k} - F^a_{12}F^a_{12}). \tag{3}$$

Fixing the coupling constant $g_0^2/V^{(3)} = 1$, we take action (3) as the action of the light-cone $SU(n)$ Yang–Mills mechanics with the Lagrangian

$$L := \frac{1}{2}(F^a_{+-}F^a_{+-} + 2F^a_{+k}F^a_{-k} - F^a_{12}F^a_{12}). \tag{4}$$

Lagrangian (4) is written in terms of the following light-cone components of the field-strength tensor in the light-front metric:

$$F^a_{+-} := \frac{\partial A^a_-}{\partial x^+} + \mathrm{f}^{abc}A^b_+A^c_-, \qquad F^a_{+k} := \frac{\partial A^a_k}{\partial x^+} + \mathrm{f}^{abc}A^b_+A^c_k,$$

$$F^a_{-k} := \mathrm{f}^{abc}A^b_-A^c_k,$$

$$F^a_{ij} := \mathrm{f}^{abc}A^b_iA^c_j, \quad i,j,k = 1,2,$$

where f^{abc} are the structure constants of $SU(n)$. Lagrangian (4) defines the light-cone $SU(n)$ Yang–Mills mechanics with $4(n^2-1)$ degrees of freedom $A_\pm$, A_k, and the evolution paramter $\tau := x^+$, the light-front time. Because the Yang–Mills theory is gauge invariant and because the instant-time states in the light-front dynamics are given at the light-cone characteristics, not all of the equations of motion are of the second order in τ (see, e.g., the discussion in [3], [24]). In other words, some of the Euler–Lagrange equations that follow from (4) define constraints in the configuration space. In the Hamiltonian description, this can be seen as follows. The Legendre transformation gives the momentum π^-_a canonically conjugate to A^a_-,

$$\pi^-_a := \frac{\partial L}{\partial \dot{A}^a_-} = \dot{A}^a_- + \mathrm{f}^{abc}A^b_+A^c_-.$$

But the equations for the momenta π^+_a and π^k_a canonically conjugate to A^a_+ and A^a_k lead to the set of primary constraints

$$\varphi^{(1)}_a := \pi^+_a = 0, \tag{5}$$

$$\chi^a_k := \pi^k_a - \mathrm{f}^{abc}A^b_-A^c_k = 0. \tag{6}$$

The presence of primary constraints affects the dynamics of the degenerate system. Its evolution is given by the total Hamiltonian

$$H_\mathrm{t} := H_\mathrm{c} + U_a(\tau)\varphi^{(1)}_a + V^a_k(\tau)\chi^a_k,$$

which differs from the Hamiltonian

$$H_\mathrm{c} = \frac{1}{2}\pi^-_a\pi^-_a - \mathrm{f}^{abc}A^b_+(A^c_-\pi^-_a + A^c_k\pi^k_a) + \frac{1}{2}F^a_{12}F^a_{12}$$

by a linear combination of the primary constraints with the indeterminate Lagrange multipliers $U_a(\tau)$ and $V^a_k(\tau)$.

We should verify the dynamical self-consistency of primary constraints (5) using the total Hamiltonian and the canonical Poisson brackets of the phase variables

$$\{A^a_\pm, \pi^\pm_b\} = \delta^a_b, \qquad \{A^a_k, \pi^l_b\} = \delta^l_k\delta^a_b.$$

From the requirement of conservation of the primary constraints $\varphi^{(1)}_a$, we have

$$0 = \dot{\varphi}^{(1)}_a = \{\pi^+_a, H_\mathrm{t}\} = \mathrm{f}^{abc}(A^b_-\pi^-_c + A^b_k\pi^k_c), \tag{7}$$

while the same procedure for the primary constraints χ_k^a leads to the self-consistency conditions on the Lagrangian multipliers $V_k^a(\tau)$

$$0 = \dot{\chi}_k^a = \{\chi_k^a, H_{\rm c}\} - 2{\rm f}^{abc} A_-^b V_k^c. \tag{8}$$

Consistency conditions (7) thus define the n^2-1 secondary constraints

$$\varphi_a^{(2)} := {\rm f}^{abc}(A_-^b \pi_c^- + A_k^b \pi_c^k) = 0, \tag{9}$$

which satisfy the $su(n)$ algebra,

$$\left\{\varphi_a^{(2)}, \varphi_b^{(2)}\right\} = {\rm f}^{abc}\varphi_c^{(2)}.$$

But the analysis of consistency conditions (8) is a more complicated task. First, the number of Lagrange multipliers that can be determined from (8) depends on the rank of the structure group. This follows directly from the form of the Poisson brackets of the constraints χ_i^a,

$$\{\chi_i^a, \chi_j^b\} = 2{\rm f}^{abc} A_-^c \delta_{ij}.$$

We analyzed the simplest case of rank 1 (the special unitary group $SU(2)$) in our previous papers. The constraints of the $SU(2)$ model, including separating them into the first and second classes, were analyzed in [20]–[22]. Below, we briefly present the previous results and then discuss the model with the first nontrivial rank-2 structure group, the light-cone $SU(3)$ Yang–Mills mechanics, in detail.

2.1. The structure group $SU(2)$. For the basis of the $su(2)$ algebra, we choose the Pauli matrices σ_1, σ_2, and σ_3. The structure constants are then given by the totally antisymmetric three-dimensional Levi-Civita symbol:

$${\rm f}^{abc} := \epsilon^{abc}, \qquad \epsilon^{123} = 1.$$

According to Eqs. (5) and (6), there are $(2^2-1) + (2^2-1)\times 2 = 9$ primary constraints $\varphi_a^{(1)}$ and χ_k^a. Consistency condition (8) for the primary constraints χ_k^a leads to the following picture of the constraints in the model:

1. Apart from the indicated first-class constraints (the *Abelian* π_a^+ and the *non-Abelian* $\varphi_a^{(2)}$), there are two more Abelian constraints without an analogue in the instant form of the $SU(2)$ mechanics, $\psi_k := A_-^a \chi_k^a$. We note that A_-^a is a *null vector* of the matrix $C_{ab} := \epsilon_{abc} A_-^c$ consisting of the Poisson brackets of the constraints χ_k^a.

2. The remaining four constraints from the set χ_k^a, forming an orthogonal complement to the constraints ψ_k, $\chi_{k\perp}^a := \chi_k^a - A_-^a(A_-^b \chi_k^b)$, are second class and satisfy the relations

$$\{\chi_{i\perp}^a, \chi_{j\perp}^b\} = 2\epsilon^{abc} A_-^c \delta_{ij}, \qquad \left\{\varphi_a^{(2)}, \chi_{k\perp}^b\right\} = \epsilon^{abc}\chi_{k\perp}^c.$$

Further analysis shows that other than the Gauss law constraints $\varphi_a^{(2)}$, there are no new secondary constraints in the model. Indeed, the Abelian constraints ψ_i do not create new ones,

$$\{\psi_i, H_{\rm t}\} = -A_i^a \varphi_a^{(2)} + \pi_a^- \chi_i^a + \epsilon_{abc} A_i^a A_k^b \chi_k^c \approx 0,$$

and consistency condition (8) for the constraints $\chi_{i\perp}^a$ allows determining the corresponding four Lagrange multipliers $V_\perp(\tau)$. In summary, the light-cone $SU(2)$ Yang–Mills mechanics has eight functionally independent first-class constraints $\varphi_a^{(1)}$, ψ_k, and $\varphi_a^{(2)}$ and four second-class constraints $\chi_{k\perp}^a$.

2.2. The structure group $SU(3)$. Because the rank of the $su(3)$ algebra is two, the null space of the matrix $C_{ab} = \mathrm{f}_{abc}A^c_-$ is two-dimensional.[2] As its basis, we choose the following null vectors, the first linear and the second quadratic in the coordinates,

$$e^{(1)}_a := A^a_-, \qquad e^{(2)}_a := \mathrm{d}_{abc}A^b_-A^c_-, \quad a,b,c = 1,2,\ldots,8.$$

Using the vectors $e^{(1)}_a$ and $e^{(2)}_a$, we decompose the set of $2\times(3^2-1) = 16$ primary constraints χ^a_k as

$$\chi^a_i = (\chi^a_{i\perp}, \psi_i, \varsigma_i), \quad i = 1,2, \tag{10}$$

where

$$\psi_i := e^{(1)}_a\chi^a_i, \qquad \varsigma_i := e^{(2)}_a\chi^a_i.$$

Decomposition (10) turns out to be very useful because of the special Poisson bracket relations for the decomposition components,

$$\{\chi^a_k, \psi_i\} = 0, \qquad \{\chi^a_k, \varsigma_i\} = 0, \qquad \{\psi_i, \varsigma_k\} = 0, \qquad \{\psi_i, \psi_j\} = 0, \qquad \{\varsigma_i, \varsigma_k\} = 0.$$

Consistency conditions (8) allow finding the Lagrange multipliers $V^a_{k\perp}$ corresponding to decomposition (10) and lead to the equalities modulo the primary constraints

$$\{\psi_i, H_\mathrm{t}\} = -A^a_i\varphi^{(2)}_a + \text{primary constraints}, \tag{11}$$

$$\{\varsigma_i, H_\mathrm{t}\} = \mathrm{d}_{abc}A^a_iF^b_{-k}F^c_{-k} - 2\mathrm{d}_{abc}A^a_-A^b_i\varphi^{(2)}_c + \text{primary constraints}. \tag{12}$$

According to (11), the constraints ψ_i do not yield new secondary constraints. But analysis of (12) shows that there are two new secondary constraints

$$\zeta_i = \mathrm{d}_{abc}A^a_iF^b_{-k}F^c_{-k}. \tag{13}$$

The constraints ζ_i commute, $\{\zeta_i, \zeta_j\} = 0$, and satisfy the relations

$$\begin{aligned} \{\psi_i, \zeta_j\} &= \delta_{ij}\mathrm{d}_{abc}A^a_-\left(F^b_{-k}\chi^c_k - \frac{1}{2}A^b_-\varphi^{(2)}_c\right), \\ \{\varsigma_i, \zeta_j\} &= -\delta_{ij}\mathrm{d}_{abc}\mathrm{d}_{cpq}A^a_-A^b_-F^p_{-k}F^q_{-k}. \end{aligned} \tag{14}$$

Further analysis using the technique of Gröbner bases shows that the right-hand side of (14) is nonvanishing modulo all known constraints and leads to the absence of tertiary constraints. The consistency condition[3]

$$\{\zeta_i, H_\mathrm{t}\} \overset{\Sigma_2}{=} \{\zeta_i, H_\mathrm{c}\} + \{\zeta_i, \varsigma_k\}V^\varsigma_k = 0$$

allows deteriming the two unknown functions V^ς_k in the decomposition of the Lagrange multipliers $V^a_k = (V^a_{k\perp}, V^\psi_k, V^\varsigma_k)$.

In summary, the complete set of constraints in the $SU(3)$ Yang–Mills mechanics is

a. π^+_a, $\varphi^{(2)}_a$, and ψ_k (18 first-class constraints) and

b. $\chi^a_{k\perp}$, ς_k, and ζ_k (16 second-class constraints).

These results are based on a tedious calculation of the Poisson bracket relations and their subsequent decomposition with respect to the complete set of constraints. For this, we used the especially constructed Gröbner basis briefly described in the appendix.

[2]The structure constants f_{abc} and d_{abc} used here correspond to the standard Gell-Mann basis for the $su(3)$ algebra.
[3]The symbol Σ_2 here denotes the constraint manifold defined by the primary and secondary constraints.

3. Local symmetries

The presence of two new first-class constraints ψ_i poses the question of the existence of a new local invariance in the model in addition to the expected residual gauge symmetry related to the initial group of isotopic $SU(n)$ rotations. To find this additional symmetry, we now pass to the problem of constructing the generator of an infinitesimal local symmetry transformation. We restrict our consideration to the case $n = 2$. The local symmetries are generated by first-class constraints (cf. [25]), but the presence of the second-class constraints in the theory seriously complicates the task of constructing the symmetry transformations in explicit form. To circumvent this difficulty, we proceed as follows. First, we effectively eliminate the second-class constraints by introducing the Dirac bracket. This means that in what follows, all expressions are evaluated modulo the second-class constraints.[4] Then, according to the method in [26] based on the abovementioned Dirac conjecture, we represent the generator G of local transformations as a linear combination of all first-class constraints,

$$G = \sum_{a=1}^{3} \varepsilon_a^{(1)} \varphi_a^{(1)} + \sum_{i=1}^{2} \eta_i \psi_i + \sum_{a=1}^{3} \varepsilon_a^{(2)} \varphi_a^{(2)} \tag{15}$$

with the eight time-dependent functions $\varepsilon_a^{(1)}(\tau)$, $\varepsilon_a^{(2)}(\tau)$, and $\eta_i(\tau)$. Generator (15) is conserved on the first-class constraint surface,

$$\frac{dG}{d\tau} \overset{\Sigma_1}{=} 0. \tag{16}$$

Consequently, not all functions $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, and η_i are independent. Indeed, evaluating the total derivative in (16), we obtain the relation

$$\dot{\varepsilon}_a^{(2)} + \varepsilon_a^{(1)} - \epsilon_{abc} \varepsilon_b^{(2)} A_+^c - \eta_i A_i^a = 0.$$

Expressing $\varepsilon_a^{(1)}$ in terms of the remaining functions $\varepsilon_a^{(2)}$ and η_i, we finally represent the generator of local transformations in the form

$$G = \left(-\dot{\varepsilon}_a^{(2)} + \epsilon_{abc} \varepsilon_b^{(2)} A_+^c + \eta_i A_i^a\right) \varphi_a^{(1)} + \eta_i \psi_i + \varepsilon_a^{(2)} \varphi_a^{(2)}. \tag{17}$$

With (17), the infinitesimal local symmetry transformations of the phase space coordinates are given by the Dirac brackets

$$\delta A_\mu = \{G, A_\mu\}_{\mathrm{D}}, \qquad \delta \pi^\mu = \{G, \pi^\mu\}_{\mathrm{D}}.$$

We will discuss all symmetry transformations and their relation to an invariance of the initial Yang–Mills theory in detail in a separate publication. Here, we only want to briefly describe the new local symmetry in the light-cone Yang–Mills mechanics, which has no analogue in the instant form. Our result is that light-cone Lagrangian (4) is invariant, i.e., $\delta_G L_{\mathrm{lc}} = 0$, under five-parameter local symmetry transformations of two different sorts:

1. the local isotopic $SU(2)$ rotations

$$\delta_\varepsilon A_+^a = \dot{\varepsilon}_a(\tau) - \epsilon_{abc} \varepsilon_b(\tau) A_+^c, \qquad \delta_\varepsilon A_-^a = -\epsilon_{abc} \varepsilon_b(\tau) A_-^c, \qquad \delta_\varepsilon A_i^a = -\epsilon_{abc} \varepsilon_b(\tau) A_i^c;$$

2. the new local nonisotopic variations of the form

$$\delta_\eta A_+^a = \eta_i(\tau) A_i^a, \qquad \delta_\eta A_-^a = 0, \qquad \delta_\eta A_i^a = \eta_i(\tau) A_-^a.$$

Having the generator of local transformations, we can now pose the question of finding a set of suitable coordinates some of which represent the invariants of these transformations. Solving this problem means explicitly reducing the Hamiltonian system onto the constraint manifold. In the next section, we discuss some of the results obtained in this direction for the simple model with the structure group $SU(2)$.

[4] This highly nontrivial operation can also be implemented using the Gröbner basis.

4. Integrability of the light-cone mechanics

After elimination of the gauge degrees of freedom, the instant $SU(2)$ Yang–Mills mechanics is a (2×6)-dimensional nondegenerate Hamiltonian system. Its Hamiltonian corresponds to the so-called Euler–Calogero–Moser many-particle system of type ID_3 in a fourth-order external potential[5]

$$H_{\mathrm{I}} := \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{2}\sum_{\text{cyclic}} \xi_a^2 \left[\frac{1}{(x_b - x_c)^2} + \frac{1}{(x_b + x_c)^2}\right] + \frac{1}{2}\sum_{a<b} x_a^2 x_b^2,$$

where (x_a, p_a) are the canonical coordinates of the three particles, $\{x_a, p_b\} = \delta_{ab}$, and the variables ξ_a describe a spin system satisfying the $so(3)$ algebra, $\{\xi_a, \xi_b\} = \varepsilon_{abc}\xi_c$.

A test of the integrability of this system indicates a complex chaotic behavior of the classical trajectories (see the discussion and references in [7]–[9]). Is the same true for the light-cone zero-mode dynamics? Here, we briefly discuss the negative answer to this question, referring to [21], where it was shown that in contrast to the instant form, the light-cone Yang–Mills mechanics reduces to the free motion of one nonrelativistic particle.

Following [21], we briefly describe the stages of the Hamiltonian reduction. To write the reduced form of the light-cone $SU(2)$ Yang–Mills mechanics explicitly, we use a compact notation, introducing the 3×3 matrix A_{ab} constructed from columns: $A := \|A_1^a, A_2^a, A_-^a\|$. Eliminating the local degrees of freedom associated with the three constraints $\varphi_a^{(2)}$ is achieved using the *polar representation* of the matrix A: $A = OS$, where S is a positive-definite symmetric 3×3 matrix and O is an orthogonal matrix. The three angles parameterizing it are gauge degrees of freedom associated with the constraints $\varphi_a^{(2)}$. To find such cyclic coordinates associated with the constraints ψ_1 and ψ_2, we represent the matrix S with respect to the principal axes,

$$S = R^{\mathrm{t}}(\chi_1, \chi_2, \chi_3)\,\mathrm{diag}(q_1, q_2, q_3) R(\chi_1, \chi_2, \chi_3),$$

using an orthogonal matrix R depending on the three Euler angles. An analysis shows that the two of them, for example, the angles χ_1 and χ_2, can be identified with the remaining pure gauge degrees of freedom.

Further, solving for the four second-class constraints $\chi_{i\perp}^a$ leads to a nondegenerate system representing a one-dimensional *free particle* or, if we allow complex solutions of these constraints (see [21] for the details), to a more interesting model, the so-called conformal mechanics. In the latter case with complex solutions of the second-class constraints taken into account, the reduced Hamiltonian has the form

$$H_{\mathrm{lc}} = \frac{1}{2}\left(p_1^2 + \frac{\kappa^2}{q_1^2}\right), \tag{18}$$

where $\kappa^2 = (p_{\chi_3}/2)^2$. Because the angle χ_3 is cyclic, the conjugate momentum p_{χ_3} is a constant of motion, and Hamiltonian (18) indeed describes the conformal mechanics with the "coupling constant" κ^2.[6]

5. Comments

We have considered the light-cone $SU(n)$ Yang–Mills field theory supposing that the gauge potentials in the classical action are functions of only the time. The dynamics of such zero modes differs significantly from the instant-time Yang–Mills mechanics. The light-cone mechanics has a more complicated description as a constrained system. Using the Dirac formalism for degenerate Hamiltonian systems, we found that in addition to the constraints generating the expected homogeneous $SU(n)$ gauge group transformations,

[5] See [11], [14] for the details of the Hamiltonian reduction.

[6] The quantity κ is the parameter measuring the deviation from the real trajectories. They are all in the subspace with $\det\|A\| = 0$ and correspond to a free particle motion.

there is a new set of first- and second-class constraints. Moreover, it turned out that both the number and the type of constraints depends on the rank of the structure group. Thus, for example, there is an additional pair of second-class constraints for $SU(3)$ in comparison with the $SU(2)$ case. In comparison, we note that in the instant form, there are only primary constraints $\varphi_a^{(1)}$ and secondary constraints $\varphi_a^{(2)}$ of the first class for all groups $SU(n)$. Because of the presence of the new constraints, the light-cone $SU(2)$ mechanics has an essential decrease in the number of "true" degrees of freedom, which finally results in the classical integrability of the model. Whether the last statement holds for the light-cone $SU(n)$ Yang–Mills mechanics with $n > 2$ is an open question.

Appendix: Description of the Gröbner basis for the light-cone $SU(3)$ Yang–Mills mechanics

The standard Dirac–Bergmann procedure for determining and classifying constraints was implemented computationally via the Gröbner bases method [16]–[18] first in Maple [19], [23] for the light-cone mechanics with the structure group $SU(2)$ using the built-in function `GroebnerBasis` with the monomial order `DegreeReverseLexicographic`. But for the group $SU(3)$, because of the substantial increase in the number of nonzero structure constants f_{abc} and d_{abc}, the computer memory turned out to be insufficient. Therefore, a special program was written in the computer algebra system *Mathematica* for constructing a homogeneous Gröbner basis (Sec. 10.2 in [16]) step by step in accordance with chosen grading variables.

We introduce the grading Γ, giving the following weights of the variables π_a^μ and A_μ^a:

$$\Gamma(\pi_a^\mu) = 2, \qquad \Gamma(A_\mu^a) = 1, \quad a = 1, 2, \ldots, 8, \quad \mu = -, 1, 2.$$

The constraints χ_k^a, $\varphi_a^{(2)}$, and ζ_i (see the respective (6), (9), and (13)), now representing the set of Γ-homogeneous polynomials, are given in Table 1 with the corresponding Γ-degree indicated.

Table 1

Γ-degree	Constraints$(i, k = 1, 2)$
2	$\chi_k^a = \pi_a^k - \mathrm{f}_{abc} A_-^b A_k^c$
3	$\varphi_a^{(2)} = \mathrm{f}_{abc}(A_-^b \pi_c^- + A_k^b \pi_c^k)$
5	$\zeta_i = \mathrm{d}_{abc}\mathrm{f}_{bpq}\mathrm{f}^{cst} A_i^a A_-^p A_k^q A_-^s A_k^t$

Further, we choose a Γ-compatible graded lexicographic order, ensuring a minimal initial number of S-polynomials, in the form

$$\pi_a^- \succ \pi_b^1 \succ \pi_c^2 \succ A_-^a \succ A_1^b \succ A_2^c, \quad a, b, c = 1, 2, \ldots, 8,$$

and

$$\pi_a^\mu \succ \pi_b^\mu \succ A_\mu^a \succ A_\mu^b, \quad \text{if } a < b$$

in the case of an identical spatial index μ. With this ordering, the constraints χ_k^a and $\varphi^{(2)}$ form the lowest homogeneous Gröbner basis components G_2 and G_3 of the respective degrees two and three. Higher-degree components are constructed step by step in the order of increasing degree according to the algorithm:

a. compute the S-polynomials for the elements of G_i and G_j: (G_i, G_j);

b. eliminate superfluous S-polynomials according to the Buchberger criteria [16]–[18];

c. compute the normal forms of the remaining S-polynomials modulo the lower-degree elements previously found.

The computational results are shown in Table 2, where we indicate the number of S-polynomials with a nonzero normal form in the component G_n and the pairs of components from which the elements are taken for forming those S-polynomials.

Table 2

G_n	Polynomials	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	8	$\varphi_a^{(2)}$
G_4	15	(G_3, G_3)
G_5	14	ζ_i, (ζ_i, G_j), $i = 1, 2$, $j = 2, 3, 4$ (G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)
G_6	13	(G_2, G_5), (G_3, G_5), (G_4, G_5), (G_5, G_5) (G_3, G_4), (G_4, G_4)

The program was written in the language of the computer algebra system *Mathematica* (version 5.0), and computations were performed on a machine with the processor 2xOpteron-242 (1.6 GHz) with 6 Gbytes of RAM. We note that the presented elements are only those lowest components of the Gröbner basis needed for this work. Further computations were not done because of the rapid increase in the required time when moving from degree to degree: it is approximately an hour for G_4, 1.5 days for G_5, and already a month for G_6.

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