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On the Homogeneous Gröbner Basis for Tensors

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Abstract—Algorithmic methods of commutative algebra based on the involutive and Gröbner bases technique are efficient means for completion of equations governing dynamical systems to involution. At the same time, when working with high-dimensional tensor quantities, direct use of standard functions for calculating Gröbner bases, which are built in computer algebra systems *Maple* and *Mathematica*, requires much memory. However, being multilinear forms, tensors admit special grading that makes it possible to classify polynomials in terms of their degree of homogeneity. With regard to this feature, we propose to use a special homogeneous Gröbner basis, which allows us to avoid difficulties associated with large amount of computation. Such a basis is constructed step by step, as the degree of the polynomial grows. As an example, an algorithm for constructing the homogeneous basis in a finite-dimensional Hamiltonian system with many polynomial constraints (the so-called Yang–Mills mechanics) is presented.

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INTRODUCTION

Completion of a system of polynomial and differential equations to involution [1, 2] is a necessary and extremely complicated stage in studying dynamical systems with nontrivial geometry of the configuration, or phase, space. Well-known examples of this kind are so-called degenerate Hamiltonian mechanical models [3–6]. When solving evolutionary problems for these models, it is required to find all conditions ensuring that the dynamics is developing in a symplectic subspace determined by the constraints and, then, carry out calculations modulo the ideal generated by them. This problem admits an algorithmization procedure. In particular, an algorithm for determining and classifying all constraints in degenerate polynomial Hamiltonian systems has been developed [7–11]. It is based on extensive use of a universal computer algebra method—the Gröbner bases technique [12, 13].

In many important problems of theoretical and mathematical physics, independent dynamical variables are high-dimensional objects of tensor nature. In view of this, implementations of the algorithm based on the usage of a Gröbner bases in standard computer algebra systems, such as *Maple* and *Mathematica*, require much memory and computation time and, therefore, lead to a desired result only in simplest models.

The goal of this paper is to discuss the fact that, in the problems where basic objects are multilinear objects (tensors), the so-called homogeneous Gröbner basis [12] is a more adequate construction. Note that this situation takes place in the majority of problems in theoretical physics. Such a basis can be used after introduction of a special grading Γ specified by an appropriate choice of weights of the variables. It is important to note that the homogeneous Gröbner basis is constructed step by step, as the degree of the polynomials grows. It is this fact that results in memory reduction. Besides, by virtue of the homogeneity property, it is often possible to use only a part of the constructed basis, confining ourselves to the degrees required by a particular problem (whereas the complete basis may be incomparably greater). To demonstrate effectiveness of the suggested construction, we build the homogeneous basis for a Hamiltonian mechanical system with 64 degrees of freedom, the so-called $SU(3)$ light-cone mechanics. Note that the complete set of involutive constraints for this model was not known before we computed it.

The paper consists of three sections. In the first section, the notion of Γ-grading for polynomials is defined, and specific features of construction of the Γ-homogeneous Gröbner basis and its use are discussed. Brief description of the model under study and statement of the problem solved by the algorithm based on the Gröbner basis are presented in the second section. In the third section, construction of the homogeneous basis is described in detail. In the Conclusions, basic results of the work are summarized.

1. SPECIFIC FEATURES OF Γ-HOMOGENEOUS GRÖBNER BASIS

In this section, we briefly remind the concept of Γ-grading for polynomials [12], specific features of construction of the Γ-homogeneous Gröbner basis, and its use. As an example, we use the ring of polynomials $K[p, q]$ in two variables p and q over a field K.

Γ-grading is a mapping of a set of monomials $T(p, q)$ into the set of positive integers $\mathbb{N}$. For example, for a monomial $p^s q^t$, the Γ-degree is given by the rule

$$\Gamma(p^s q^t) = \alpha_p s + \alpha_q t, \quad \alpha_p, \alpha_q \in \mathbb{N},$$

where α_p and α_q denote weights of variables p and q, respectively. If, for some α_p and α_q, all monomials of a given polynomial have the same Γ-degree, the polynomial is said to be Γ-homogeneous. To find weights of the variables such that each polynomial from a given set is Γ-homogeneous for some Γ-degree, we need to solve a system of linear equations [12]. Let weights of variables p and q be $\alpha_p = 2$ and $\alpha_q = 1$.

Further, a Γ-compatible ordering $\succ$ on the set of monomials $T(p, q)$ is introduced as, for example,

$$\text{(a) } deglex\ p \succ q \Rightarrow p \succ q^2,$$

or

$$\text{(b) } deglex\ q \succ p \Rightarrow q^2 \succ p.$$

Now, Γ-homogeneous polynomials can be compared with respect to their Γ-degrees. For example,

$$p^2 + q^4 \succ pq + q^3 \succ p + q^2.$$

A typical problem for the models mentioned in the Introduction is as follows. Let a subspace $M \subset \mathbb{R}^{2n}$ be given by a set F of Γ-homogeneous polynomials in the ring $K[q_i, p_j]$ $(i, j = 1, \ldots, n)$:

$$\varphi_\alpha(q_i, p_j) = 0, \quad \alpha = 1, \ldots, k.$$

It is required to find a decomposition of some polynomial $f(q_i, p_j)$ modulo M. Solution of this problem in the framework of an algorithmic approach with the use of Gröbner bases is based on the following assertion [12].

Given a finite set F of polynomials each of which is homogeneous with respect to a given grading Γ, one can calculate the Γ-homogeneous Gröbner basis of degree d of the ideal Id(F) *by means of the Buchberger algorithm by dropping all S-polynomials of degrees greater than d. The result is sufficient for checking whether an arbitrary polynomial f of degree* $\Gamma(f) \le d$ *belongs to the ideal* Id(F).

In what follows, we show that, in the problems where polynomial equations determining subspace M admit introduction of the corresponding homogeneity, it is the use of the Γ-homogeneous Gröbner basis that is most efficient from the computational standpoint.

2. EXAMPLE: LIGHT-CONE GAUGE MECHANICS

The light-cone Yang–Mills mechanics [7–9] is the Yang–Mills field theory under the assumption that the fields along light front are homogeneous. This model is a finite-dimensional Hamiltonian system with polynomial constraints the ideal of which turns out to have a quite nontrivial Gröbner basis. Dynamical variables in the model are differential 1-forms with values in the algebra of group $SU(n)$,

$$A := A^a_\mu T_a \mathrm{d}x^\mu, \quad \mu = +, -, 1, 2,$$
$$a = 1, \ldots, n^2 - 1,$$

where $\mathrm{d}x^\mu = (\mathrm{d}x^+, \mathrm{d}x^-, \mathrm{d}\mathbf{x}_\perp)$ are differentials of the coordinates taken here in the form of standard "light-cone" variables of the four-dimensional Minkowski space (see [14] for detail). The components A^a_μ have both four-dimensional Lorentz space–time index μ and the isotopic group index a; i.e., they are vectors in the corresponding spaces. The $n \times n$ matrices T_a form a basis of algebra $su(n)$ and satisfy the commutation relations

$$[T_a, T_b] = 2i f_{abc} T_c$$

with structural constants f_{abc}.

By virtue of the homogeneity assumption, the fields $A^a_\mu(x^+)$ depend only on the variable x^+, which is an evolutionary parameter on the light cone. It is this dependence of the components of the 1-form that underlines the Yang–Mills mechanics on the light front (see [15] for detail).

The Lagrangian L of the model is given by

$$L\mathrm{d}V := \mathrm{tr} F \wedge *F, \tag{1}$$

where $\mathrm{d}V$ is the 4-form of space–time volume and the right-hand side is the trace of the external product of the field strength tensor, namely, the curvature 2-form $F := dA + A \wedge A$ and the form $*F$ that is dual to F with respect to the Minkowski space metrics.

The gauge invariance of the Yang–Mills field theory and specific features of the light cone theory formulation give rise to degeneracy of the system of evolutionary equations (see [5, 14]). Namely, it turns out that not all equations of motion contain second-order derivatives with respect to the "time" x^+.

When turning to the Hamiltonian formalism by means of the Legendre transform

$$\pi^+_a := \frac{\partial L}{\partial \dot{A}^a_+} = 0,$$

$$\pi^-_a := \frac{\partial L}{\partial \dot{A}^a_-} = \dot{A}^a_- + f_{abc} A^b_+ A^c_-,$$

$$\pi_a^k := \frac{\partial L}{\partial \dot{A}_k^a} = f_{abc}A_-^b A_k^c,$$

the above-mentioned theory degeneracy reveals itself in the presence of the so-called primary constraints

$$\Sigma\colon\ \varphi_a^{(1)} := \pi_a^+ = 0,$$
$$\chi_k^a := \pi_k^a + f_{abc}A_-^b A_k^c = 0.$$

The system dynamics is given by the total Hamiltonian

$$H_T := H_C + u_a\varphi_a^{(1)} + v_k^a\chi_k^a, \quad (2)$$

which is the sum of the canonical Hamiltonian

$$H_C = \frac{1}{2}\pi_a^-\pi_a^- - f_{abc}A_+^b(A_-^c\pi_a^- + A_k^c\pi_a^k) + \frac{1}{2}V(A),$$
$$V(A) := \frac{1}{2}f_{abc}A_i^bA_j^cf_{ade}A_i^dA_j^e, \quad i, j = 1, 2,$$

and the primary constraints multiplied by arbitrary Lagrange multipliers u_a, u_k^a, which depend only on the "time" x^+. In order that the system trajectory remain all time on the surface determined by the primary constraints Σ, it is required to meet all consequences of the conditions

$$\dot{\varphi}_\alpha^{(1)} = \{\varphi_\alpha^{(1)}, H_T\} \overset{\Sigma}{=} 0,$$
$$\dot{\chi}_k^a = \{\chi_k^a, H_T\} \overset{\Sigma}{=} 0,$$

where we introduced the Poisson brackets, which take the following form for canonically conjugate variables:

$$\{A_\pm^a, \pi_b^\pm\} = \delta_b^a, \quad \{A_k^a, \pi_b^l\} = \delta_k^l\delta_b^a.$$

This situation is quite typical in the theory of degenerate dynamical systems, when it is required to carry out calculations modulo surface of all constraints Σ arising in the systems. In the case of polynomial constraints, this can be done with the help of the Gröbner bases technique.

3. HOMOGENEOUS GRÖBNER BASIS IN THE CASE OF THE *SU*(3) GROUP

In this section, we apply the grading incorporation procedure to the system described in the previous section for the case of the $SU(3)$ group. Then, we present an algorithm for construction of the homogeneous Gröbner basis and its basic parameters.

The $su(3)$ algebra is an algebra of rank 2, and its properties are given by two independent sets of coefficients: antisymmetric f_{abc} and symmetric d_{abc} structure constants. The set of these constants used in the subsequent calculations is presented in the Appendix; it corresponds to the case where the basis of the $su(3)$ algebra is given by well-known Gell–Mann λ-matrices.

Table 1. Original set of constraints

Γ-degree	Constraints ($i, k = 1, 2$)
2	$\chi_k^a = \pi_a^k - f_{abc}A_-^bA_k^c$
3	$\varphi_a^{(2)} = f_{abc}(A_-^b\pi_c^- + A_k^b\pi_c^k)$
5	$\zeta_i = d_{abc}f_{bpq}f^{cst}A_i^aA_-^pA_k^qA_-^sA_k^t$

Table 2. The homogeneous Gröbner basis up to the 6th Γ-degree

G_n	Number of elements	Constraints and S-polynomials
G_2	16	χ_k^a
G_3	8	$\varphi_a^{(2)}$
G_4	15	(G_3, G_3)
G_5	14	ζ_i, (ζ_i, G_j), $i = 1, 2, j = 2, 3, 4$ (G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)
G_6	13	(G_2, G_5), (G_3, G_5), (G_4, G_5), (G_5, G_5), (G_3, G_4), (G_4, G_4)

Construction. Let us introduce a Γ-grading by specifying the following weights of variables π_a^μ and A_μ^a:

$$\Gamma(\pi_a^\mu) = 2, \quad \Gamma(A_\mu^a) = 1,$$
$$a = 1, 2, \ldots, 8, \quad \mu = -, 1, 2. \quad (3)$$

The constraints (see [15]) that now represent the set of Γ-homogeneous polynomials are given in Table 1 with the indication of the corresponding Γ-degree.

Let us select a Γ-compatible graded lexicographical ordering that ensures the minimal initial number of S-polynomials:

$$\pi_a^- \succ \pi_b^1 \succ \pi_c^2 \succ A_-^a \succ A_1^b \succ A_2^c,$$
$$a, b, c = 1, 2, \ldots, 8, \quad (4)$$

or, in the case of identical spatial index μ,

$$\pi_a^\mu \succ \pi_b^\mu \succ A_\mu^a \succ A_\mu^b, \text{ if } a < b. \quad (5)$$

Under such ordering, the constraints χ_k^a and $\varphi^{(2)}$ form the lowest components G_2 and G_3 of the homogeneous Gröbner basis of degrees 2 and 3, respectively. The components of higher degrees are constructed successively in the increasing order of their degrees in accordance with the algorithm as follows:

(i) calculate S-polynomials for elements from G_i and G_j as (G_i, G_j);

Table 3. The homogeneous Gröbner basis for the constraints χ_k^a

G_n	Number of elements	Constraints and *S*-polynomials
G_2	16	χ_k^a
G_3	72	(G_2, G_2)
G_4	76	$(G_2, G_3), (G_3, G_3)$
G_5	376	$(G_2, G_4), (G_3, G_3), (G_3, G_4), (G_4, G_4)$

(ii) eliminate redundant *S*-polynomials using the Buchberger criteria;

(iii) calculate normal forms for the remaining *S*-polynomials with respect to the already found lowest components.

Results of calculation of the homogeneous Gröbner basis up to the 6th Γ-degree inclusively are shown in Table 2. The table also shows the number of *S*-polynomials with a nonzero normal form in the component G_n and pairs of the components from which elements used for forming these *S*-polynomials were taken.

The program was written in the language of the computer algebra system *Mathematica* (*version* 5.0), and calculations were carried out on a computer with 2xOpteron-242 (1.6 GHz) processor. The computation time grew greatly with the increase of the degree. For example, the calculation of G_4, G_5, and G_6 took one hour, one and a half days, and one month, respectively. For the *S*-polynomials of the same Γ-degree, the reduction time was also considerably different. As a rule, the nonzero *S*-polynomials turned out in the normal form at once. Most of the time was spent on nonzero reductions, which agrees with the well-known results for polynomials systems in many variables. The specific feature of our calculations was that up to 80% of time was spent on the reduction of the polynomials the tensor structure of which was clearly broken.

The amount of memory used (about 40 Kb) almost did not change in the course of the calculation. It should be noted that concrete characteristics may depend on a particular algorithm implementation.

Application. The basis was used in the framework of the Dirac–Bergman–Gröbner algorithm [7–11] for finding a complete set of constraints and classifying them by their class for the *SU*(3)-light-cone Yang–Mills mechanics. We present an example of a concrete calculation, which demonstrates efficiency of using the basis. It is required to calculate the Poisson bracket modulo the constraints given in Table 1 for the convolution

$$\psi_i = A_-^a \chi_i^a = A_-^a \pi_a^i \tag{6}$$

with the complete Hamiltonian H_T (2). Using the Gröbner bases technique, we find out that the result of calculation of

$$\{\psi_i, H_T\} = \pi_a^- \pi_a^i + f_{abc} f_{cpq} A_-^a A_j^b A_j^p A_i^q$$

is zero on the constraints Σ surface

$$\{\psi_i, H_T\} = -A_i^a \varphi_a^{(2)} + \pi_a^- \chi_i^a + f_{abc} A_i^a A_k^b \chi_k^c \overset{\Sigma}{=} 0.$$

Calculations of this kind “by hand” are extremely complicated, since they require knowledge of nontrivial identities involving many variables and structural constants. It is especially difficult to prove without using computer methods that the result does not reduce to zero. Note also that, in order to obtain the decomposition in an explicit form, it is required to know representation of each element of the Gröbner basis in terms of the original set of constraints.

Effect of lexicographical ordering. Instead of the lexicographical ordering (4), (5), we may set

$$A_1^b > A_2^c > A_-^a > \pi_b^1 > \pi_c^2 > \pi_a^-, \quad a, b, c = 1, 2, \ldots, 8, \tag{7}$$

$$A_\mu^a > A_\mu^b > \pi_a^\mu > \pi_b^\mu, \text{ if } a < b. \tag{8}$$

Table 3 shows that, under such ordering, the number of *S*-polynomials greatly increases even in the case of an incomplete set of constraints.

It is interesting that G_3 contains ψ_i(6), which are important in the Dirac–Bergman–Gröbner algorithm being generators of additional gauge transformations (see [15]). They are lacking when the dynamics is described in the instant form and have been found with the help of an absolutely independent procedure. Such a feature of the Gröbner bases is a new aspect of using standard bases technique in studying dynamical systems. On the other hand, tensor structure is not taken into account in the Buchberger algorithm; therefore, an additional modification of the calculation technique is required to improve its computational efficiency.

Comparison with the case of the *SU*(2) group. The group *SU*(*n*) has $n^2 - 1$ independent generators; hence, the number of components of matrix *A* grows considerably when turning to groups of large order. For example, in the case of *SU*(2), we deal with 12 variables, whereas the number of variables in the case of *SU*(3) is as many as 32. Besides, it turned out that a more serious difficulty from the computational standpoint is associated with the fact that, for the groups *SU*(*n*) with $n \geq 3$, there exist additional constants d_{abc}, which results in the appearance of additional invariant structures, the presence of which makes analysis of the membership in the ideal an extremely cumbersome procedure. As a consequence of this, in the case of a model with the structural group *SU*(3), the main memory of size 6 Gb turned out insufficient for the calculation of a Gröbner basis by means of standard built-in functions in the *Maple 10* and *Mathematica 5.0* systems. At the same time, in the case of $SU(2)$, the calculation of the basis by means of the `GroebnerBasis` function in the *Mathematica 5.0* system with the use of the inverse lexicographical monomial order

$$\{\pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2, \pi_3^1, \pi_3^2, \pi_1^-, \pi_2^-, \pi_3^-,$$
$$A_1^1, A_2^1, A_1^2, A_2^2, A_1^3, A_2^3, A_-^1, A_-^2, A_-^3\}$$

takes 60 s; the basis contains 64 elements and turns out automatically Γ-homogeneous according to grading (3).

4. CONCLUSIONS

Specific features of homogeneous Gröbner bases make it possible to build them by the degrees in accordance with the selected grading Γ. This considerably reduces the computation time and required memory if the maximum degree of the polynomials the membership of which in the homogeneous ideal needs to be checked is a priori known. The efficiency of using such special bases in studies of dynamical systems when completing them to involution has been demonstrated on the example of the $SU(3)$-light-cone Yang–Mills mechanics. The tensor formulation of models of this kind gives rise to a great number of variables and, as a result, to the impossibility of using standard packages for calculation of the bases. On the other hand, it is such a formulation that makes it possible to take advantage of the homogeneous Gröbner bases.

It should be noted that the main portion of the computation time was spent on zero reductions, especially for the S-polynomials with obviously disturbed tensor structure. Since the tensor structure cannot directly be taken into account in the Buchberger algorithm, the group aspects of the construction of the homogeneous Gröbner bases are of interest. This is important for the efficient use of Gröbner bases in the theory of degenerate Hamiltonian systems with symmetries.

APPENDIX

The eight traceless 3 × 3 Hermitian Gell–Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (9)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

serve as a basis of the $su(3)$-algebra and satisfy the commutation relations

$$[\lambda_a, \lambda_b] = 2\mathrm{i}\sum_{c=1}^{8} \mathrm{f}_{abc}\lambda_c \qquad (10)$$

with the structure constants f_{abc} that are antisymmetric with respect to all indices. The nonzero values are as follows:

$$\mathrm{f}_{123} = 1,$$
$$\mathrm{f}_{147} = \mathrm{f}_{246} = \mathrm{f}_{257} = \mathrm{f}_{345} = \mathrm{f}_{516} = \mathrm{f}_{637} = 1/2,$$
$$\mathrm{f}_{458} = \mathrm{f}_{678} = \sqrt{3}/2.$$

The product of any two λ-matrices can be represented as

$$\lambda_a\lambda_b = \frac{2}{3}\delta_{ab}\mathrm{I} + \sum_{c=1}^{8}(\mathrm{d}_{abc} + \mathrm{i}\mathrm{f}_{abc})\lambda_c$$

with the following nonzero symmetric structure constants d_{abc}:

$$\mathrm{d}_{118} = \mathrm{d}_{228} = \mathrm{d}_{338} = \frac{1}{\sqrt{3}},$$

$$\mathrm{d}_{146} = \mathrm{d}_{157} = \mathrm{d}_{256} = \mathrm{d}_{344} = \mathrm{d}_{355} = \frac{1}{2},$$

$$\mathrm{d}_{247} = \mathrm{d}_{366} = \mathrm{d}_{377} = -\frac{1}{2},$$

$$\mathrm{d}_{448} = \mathrm{d}_{558} = \mathrm{d}_{668} = \mathrm{d}_{778} = -\frac{1}{2\sqrt{3}},$$

$$\mathrm{d}_{888} = -\frac{1}{\sqrt{3}}.$$

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